Transforms Part 2 & Image Processing

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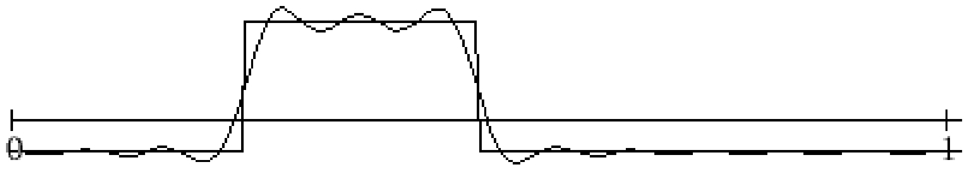
ANY function f(x) can be represented exactly as a sum of sin() functions with specific amplitudes and phases.

Fourier Representation



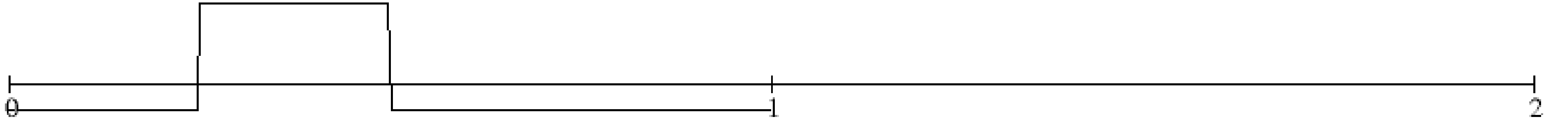
Homework

- Compute the continuous, infinite Fourier Transform of:

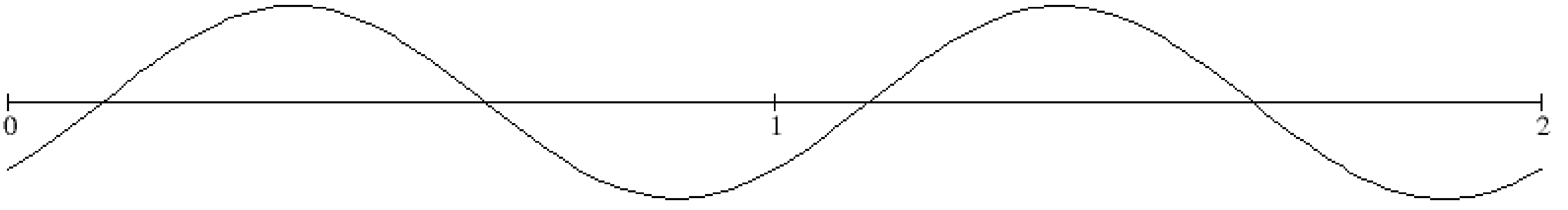
$$f(x) = \begin{matrix} 0 < x < 1 : f(x) = 1.0 \\ otherwise : f(x) = 0 \end{matrix}$$

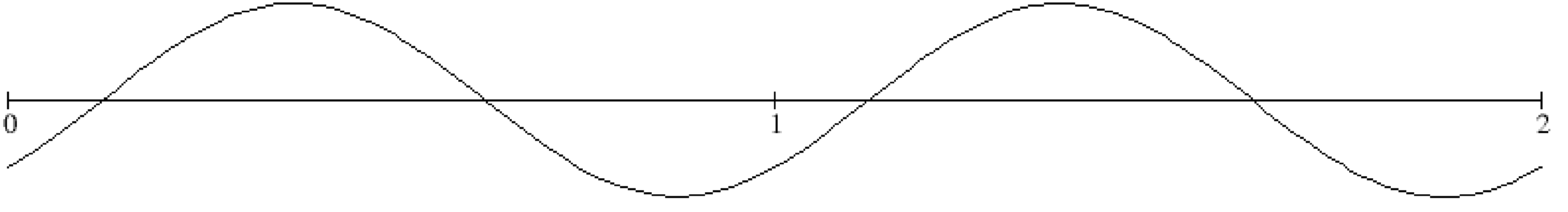
- Now say the function is finite, defined on the range 0<x<2. The transform is the same, of course, but the result is only defined for specific k values (see the lecture notes). Plot the sum of the first 10 elements of this Fourier series on the range 0<x<6. (turn in the plot) what do you observe ? What would be different if you used 100 elements ? What would be the same ?

Fourier Representation

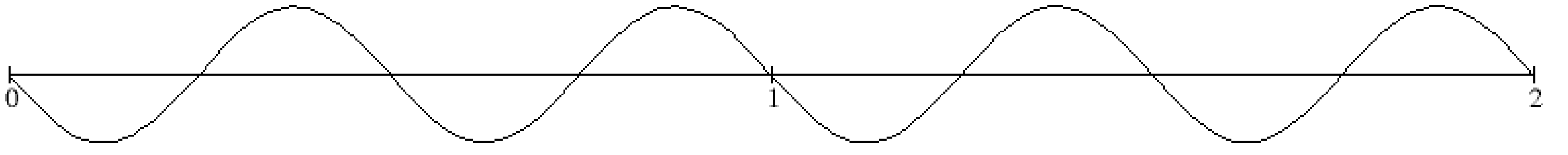


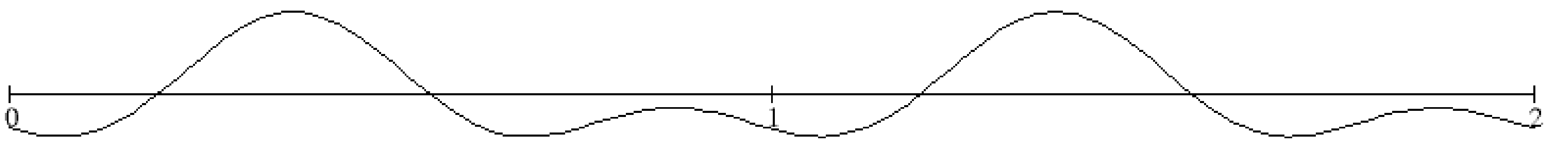
Fourier Representation



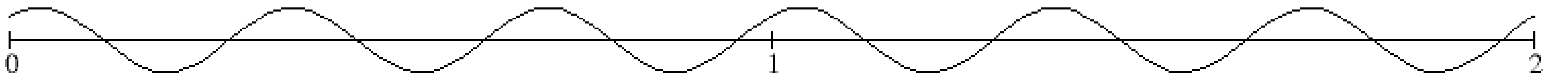


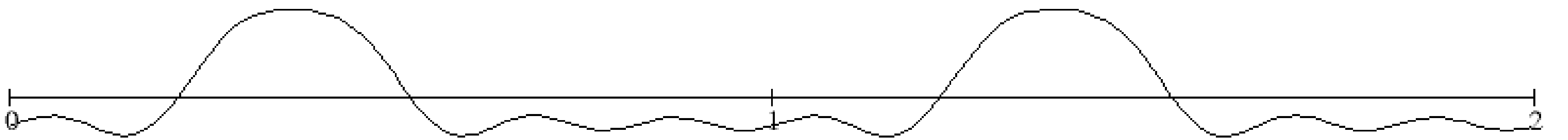
Fourier Representation





Fourier Representation





Fun With Lasers II

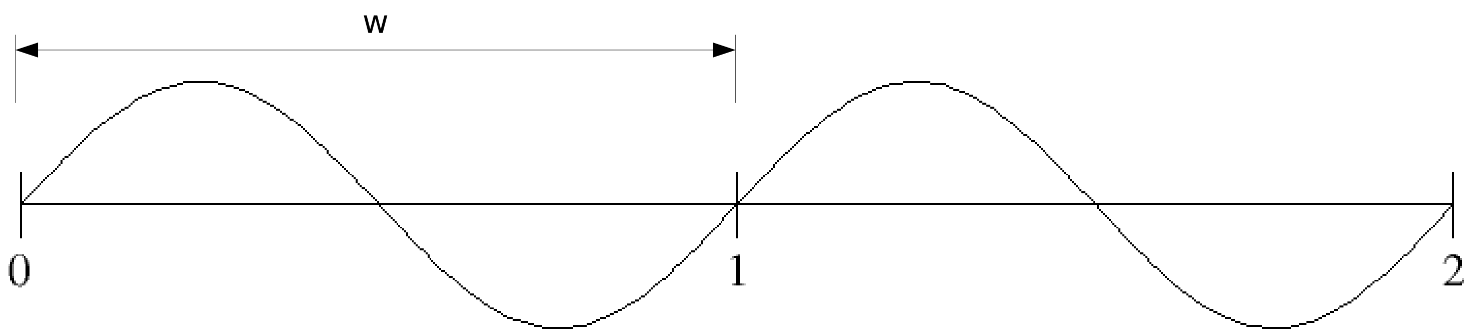
Fourier Transforms

$$\bar{F}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(k) e^{ikx} dk$$

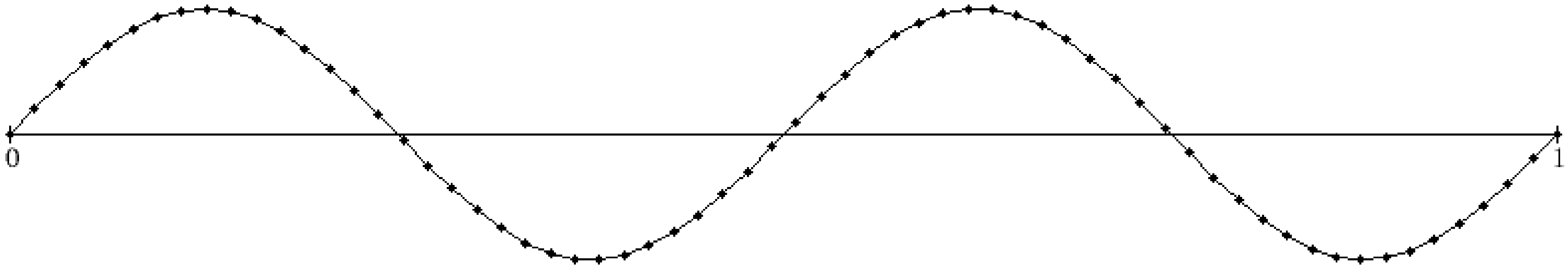
Finite Fourier Transforms

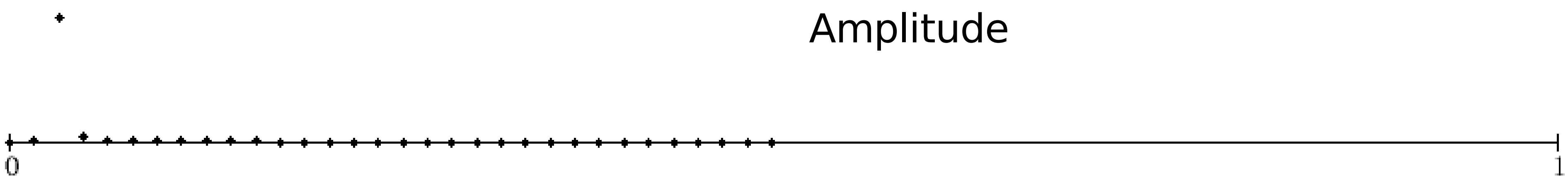
$$\bar{F}_k = \int_0^w f(x) e^{-i(2\pi k/w)x} dx$$



$$k \subset [integers]$$

Finite Discrete Fourier Transforms





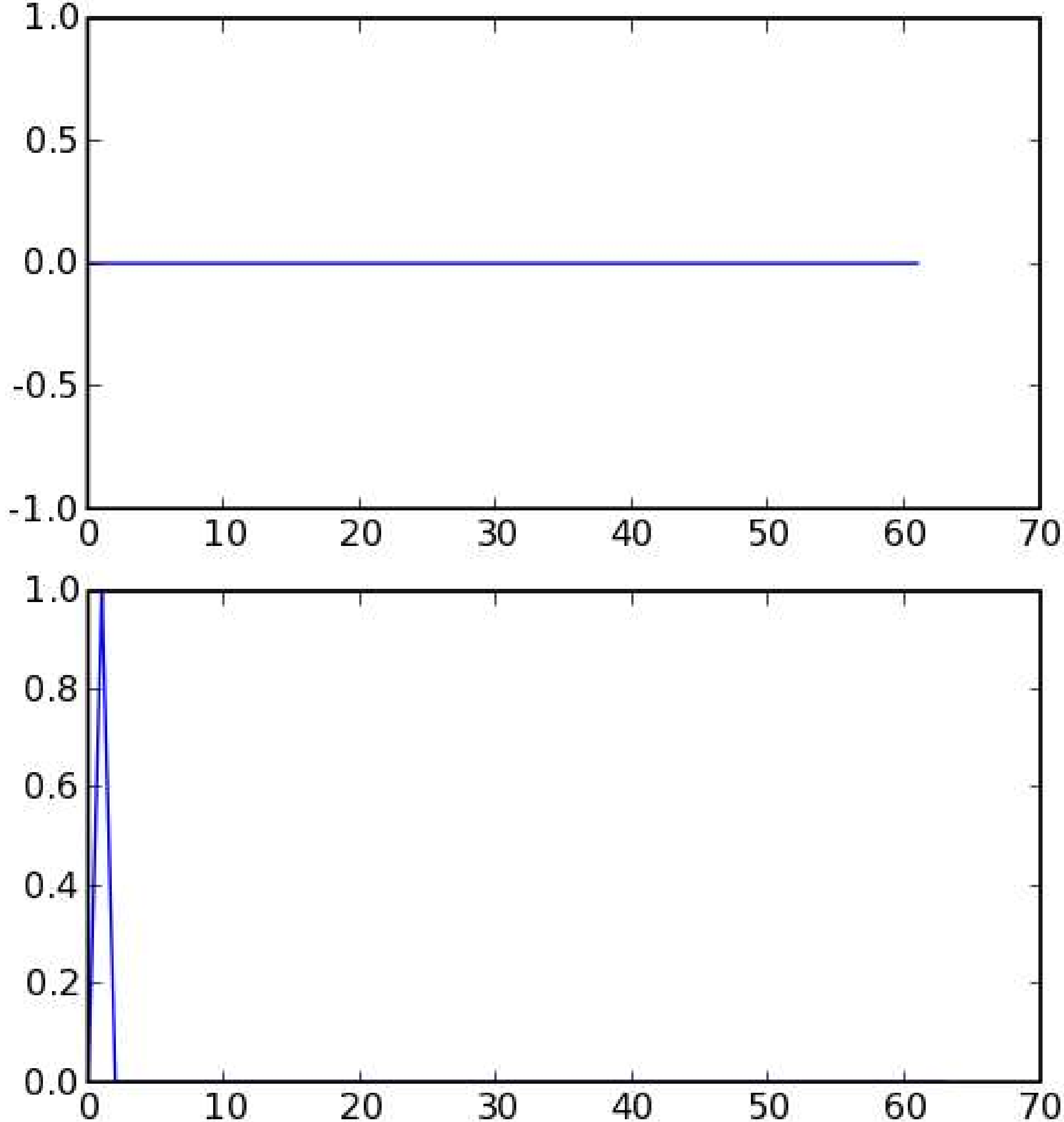
$$\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-i(2\pi k/w)x}$$

Finite range in real space -> discrete in Fourier space
Finite range in Fourier space -> discrete in Real space
Finite range implies periodicity in the same space

$$\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-ikx} \rightarrow$$

$$\begin{bmatrix} \bar{F}_0 & \bar{F}_1 & \bar{F}_2 & \bar{F}_3 \end{bmatrix} = \begin{bmatrix} e^{-ik_0x_0} & e^{-ik_0x_1} & e^{-ik_0x_2} & e^{-ik_0x_3} \\ e^{-ik_1x_0} & e^{-ik_1x_1} & e^{-ik_1x_2} & e^{-ik_1x_3} \\ e^{-ik_2x_0} & e^{-ik_2x_1} & e^{-ik_2x_2} & e^{-ik_2x_3} \\ e^{-ik_3x_0} & e^{-ik_3x_1} & e^{-ik_3x_2} & e^{-ik_3x_3} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

FFT



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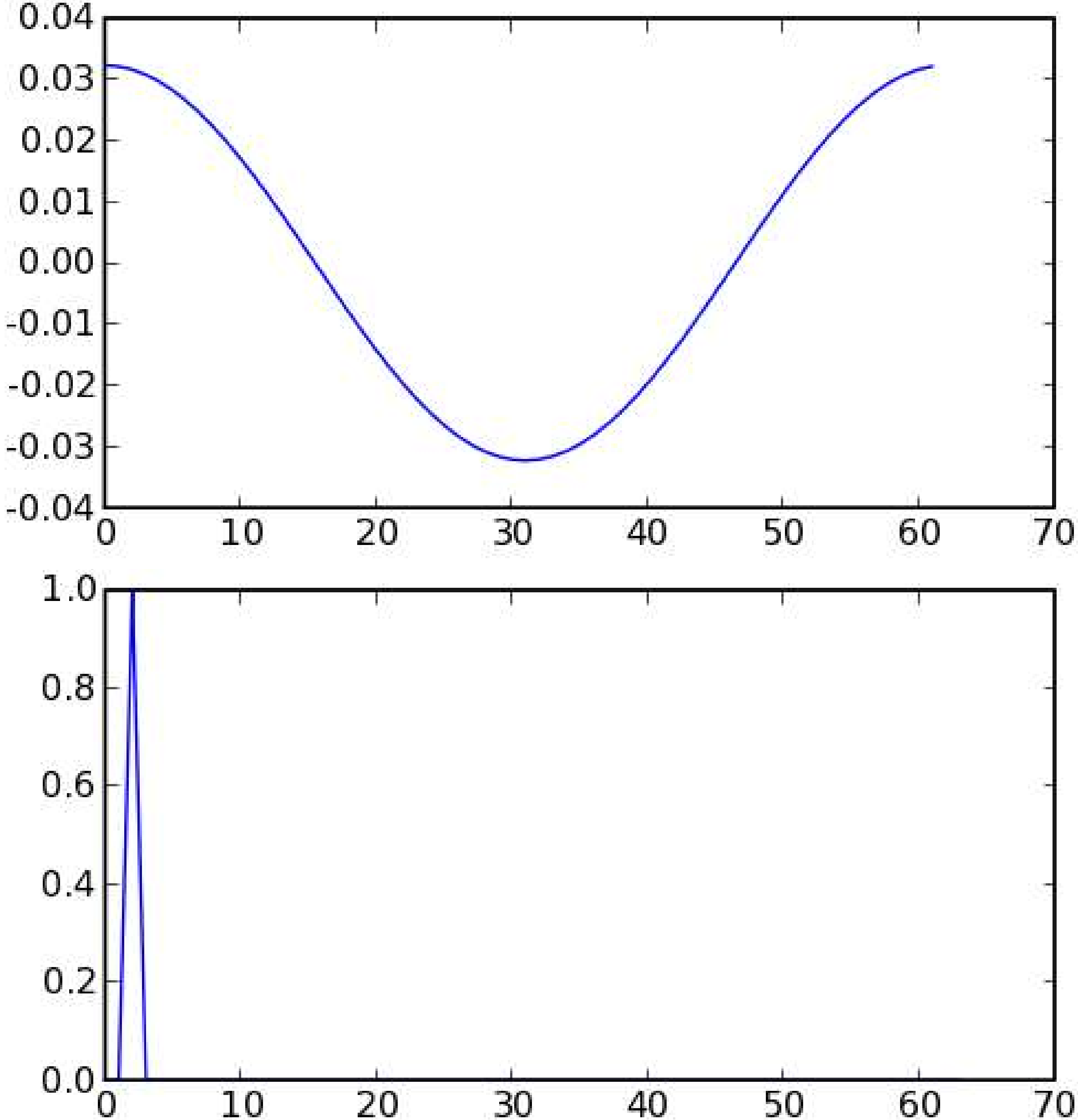

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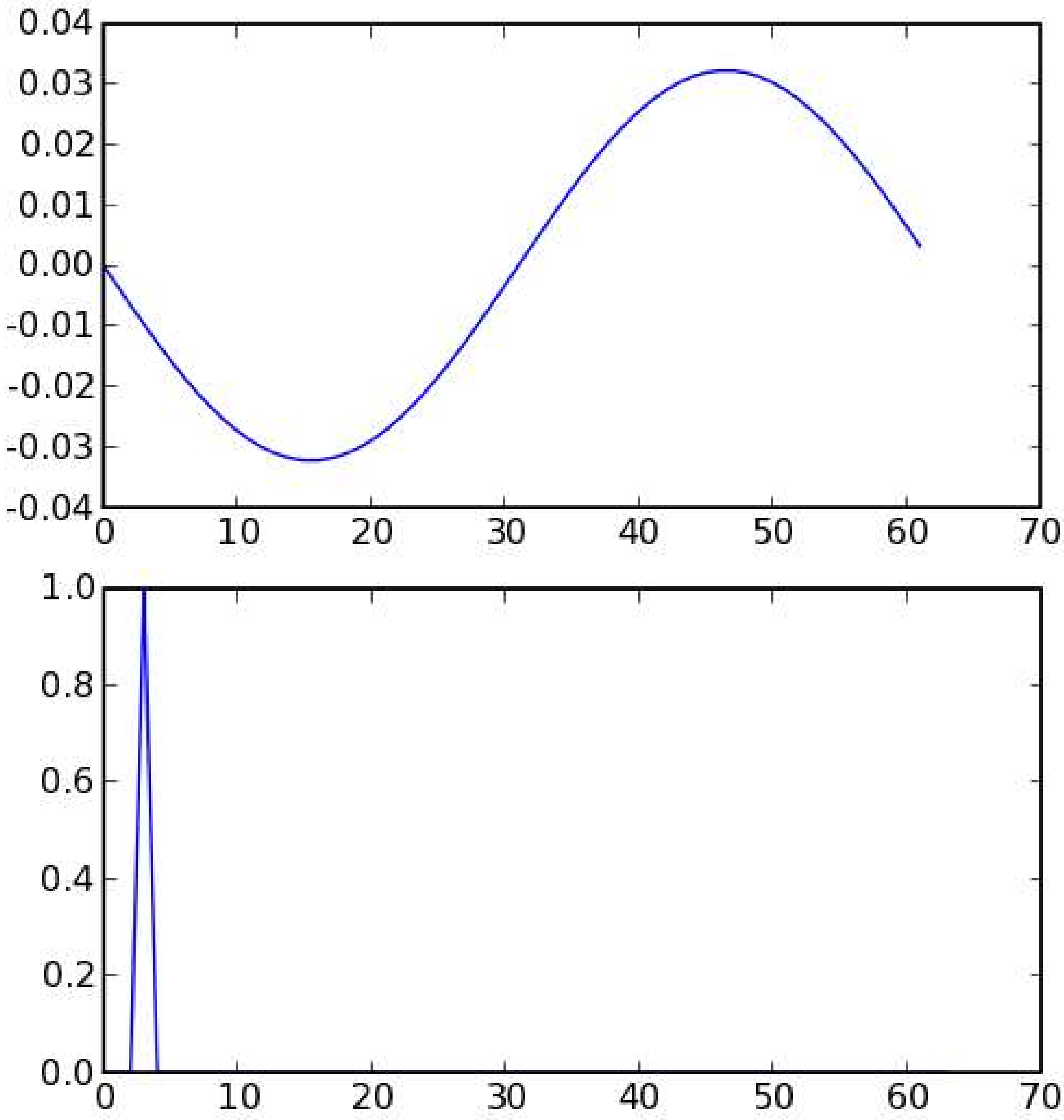

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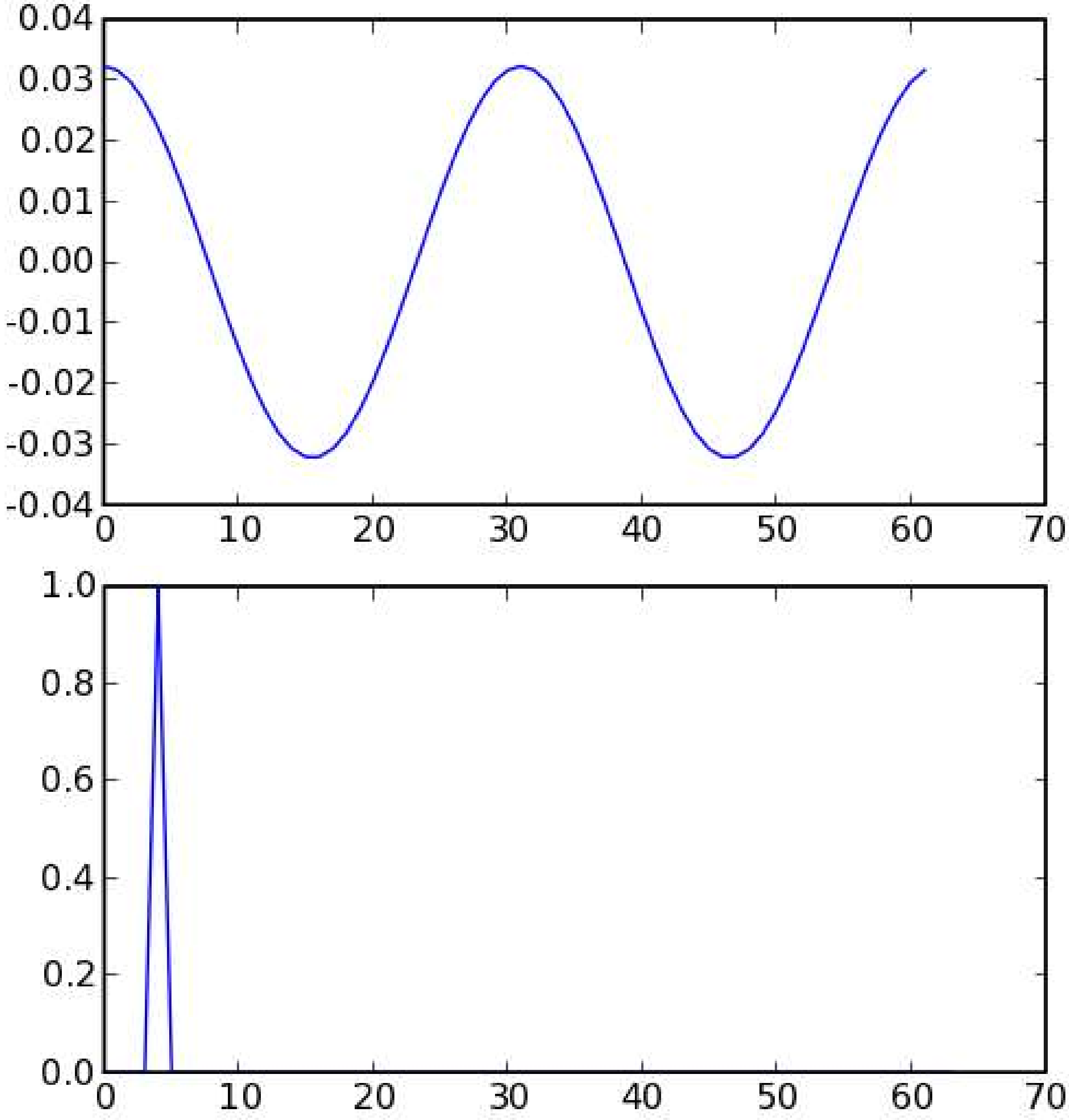

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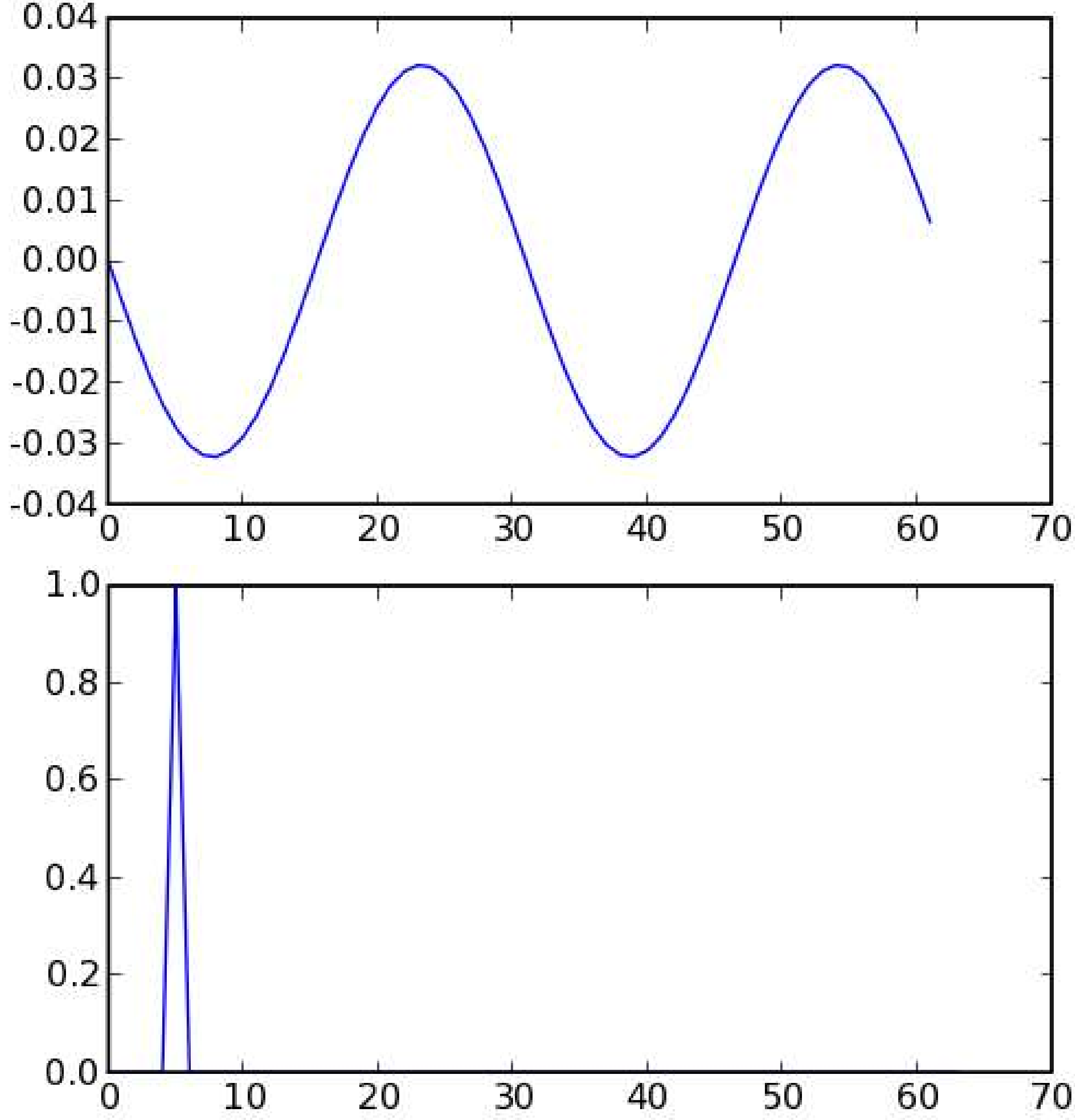
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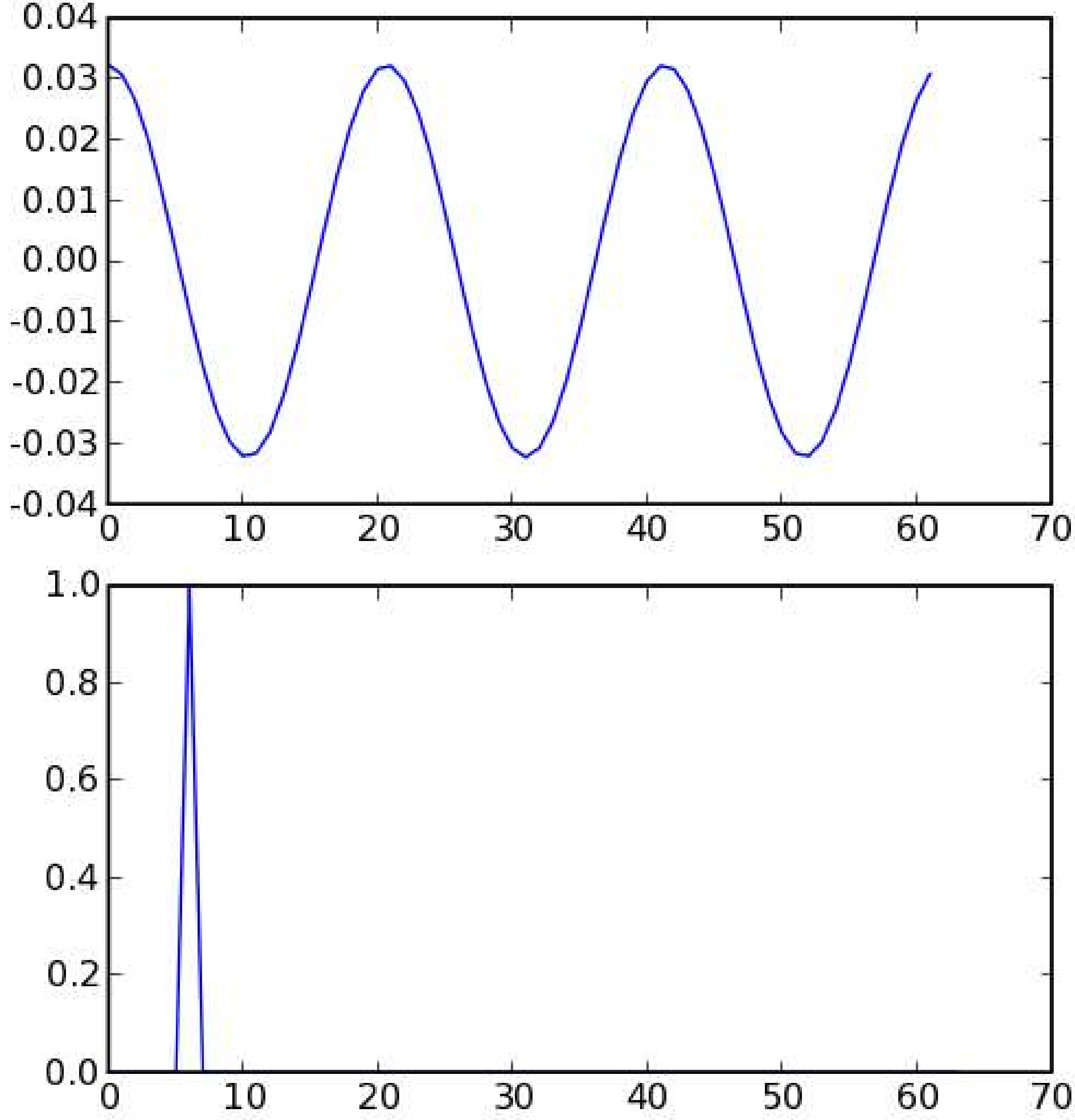
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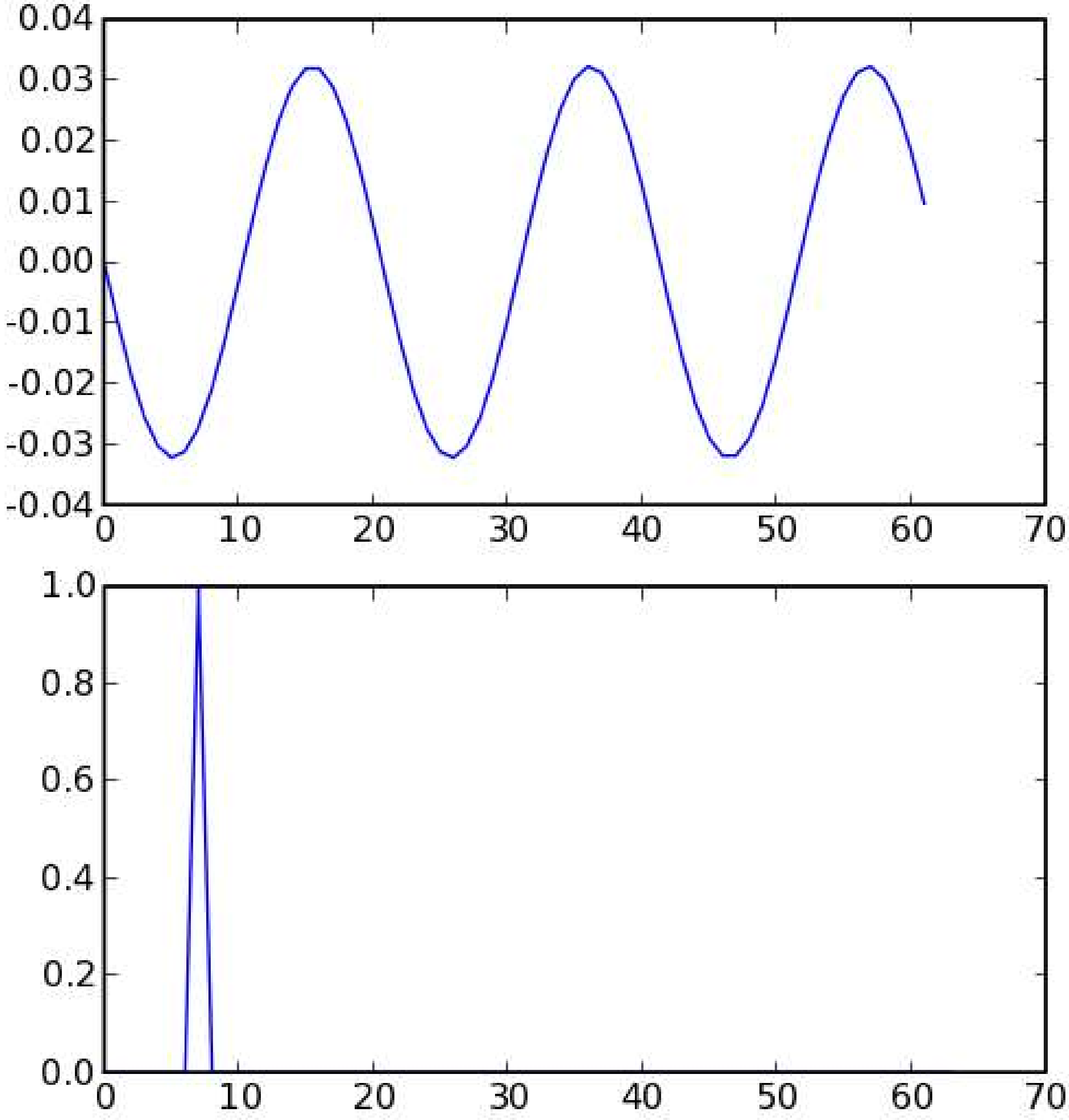

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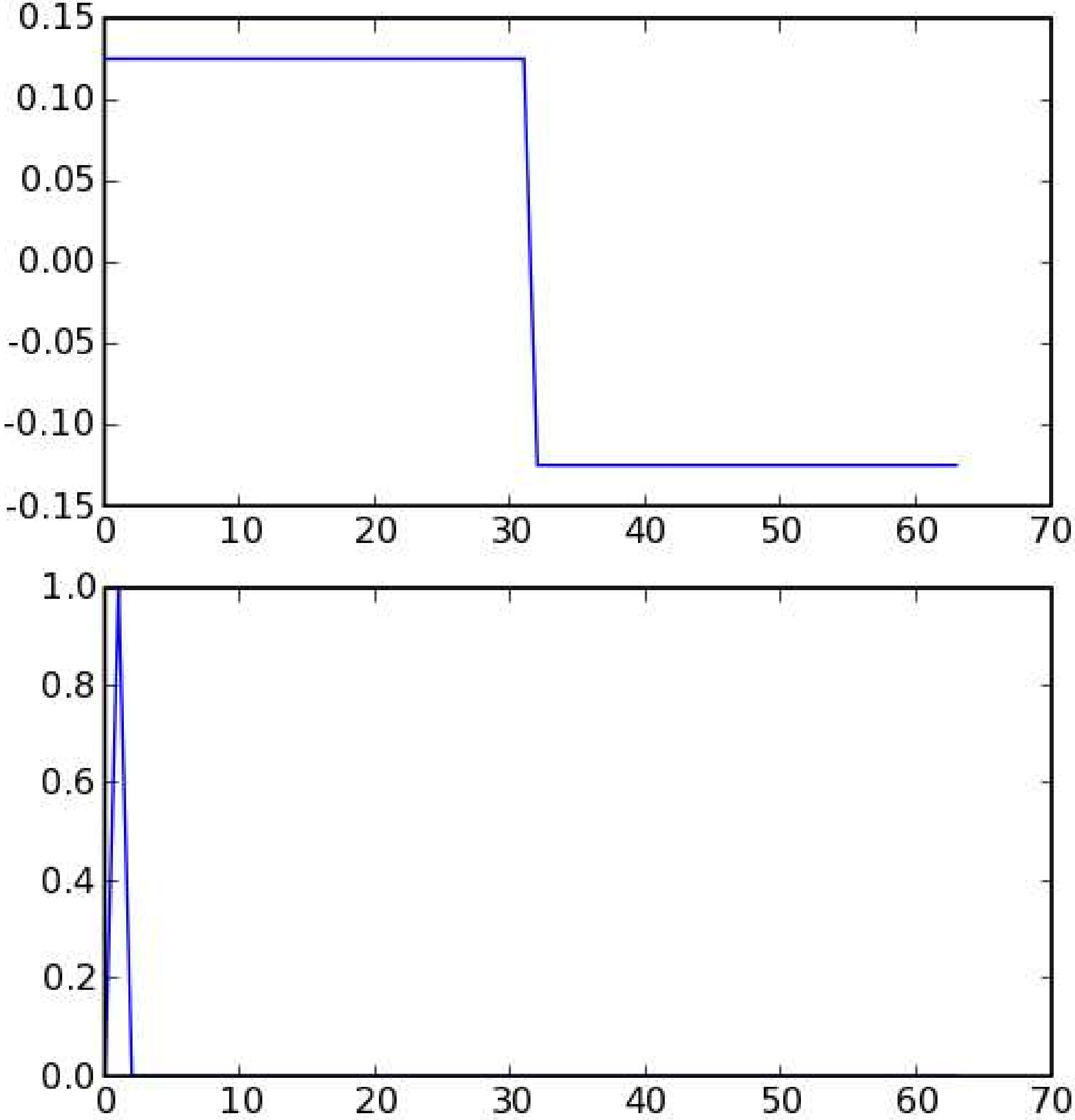
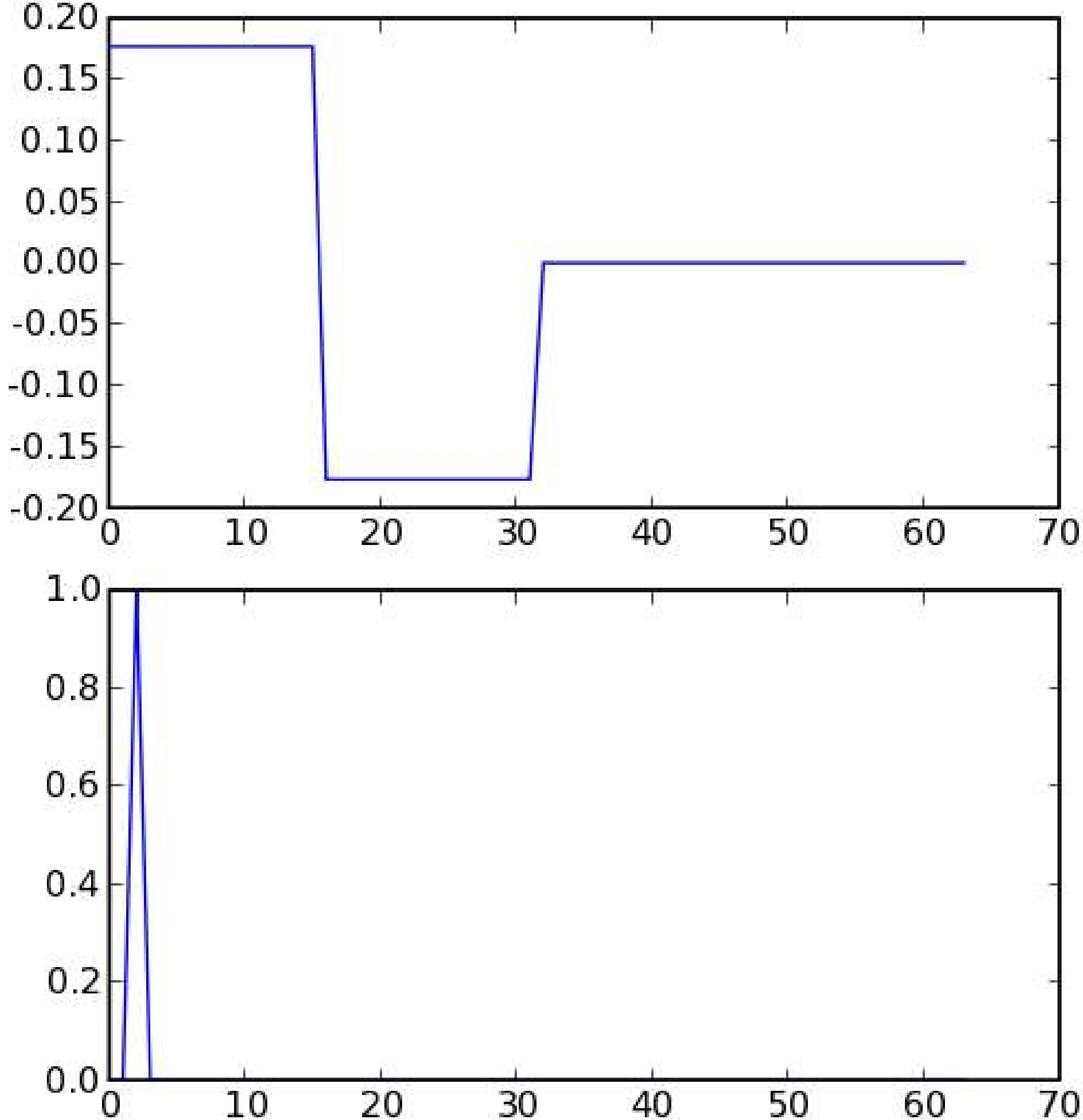
Harr Wavelet

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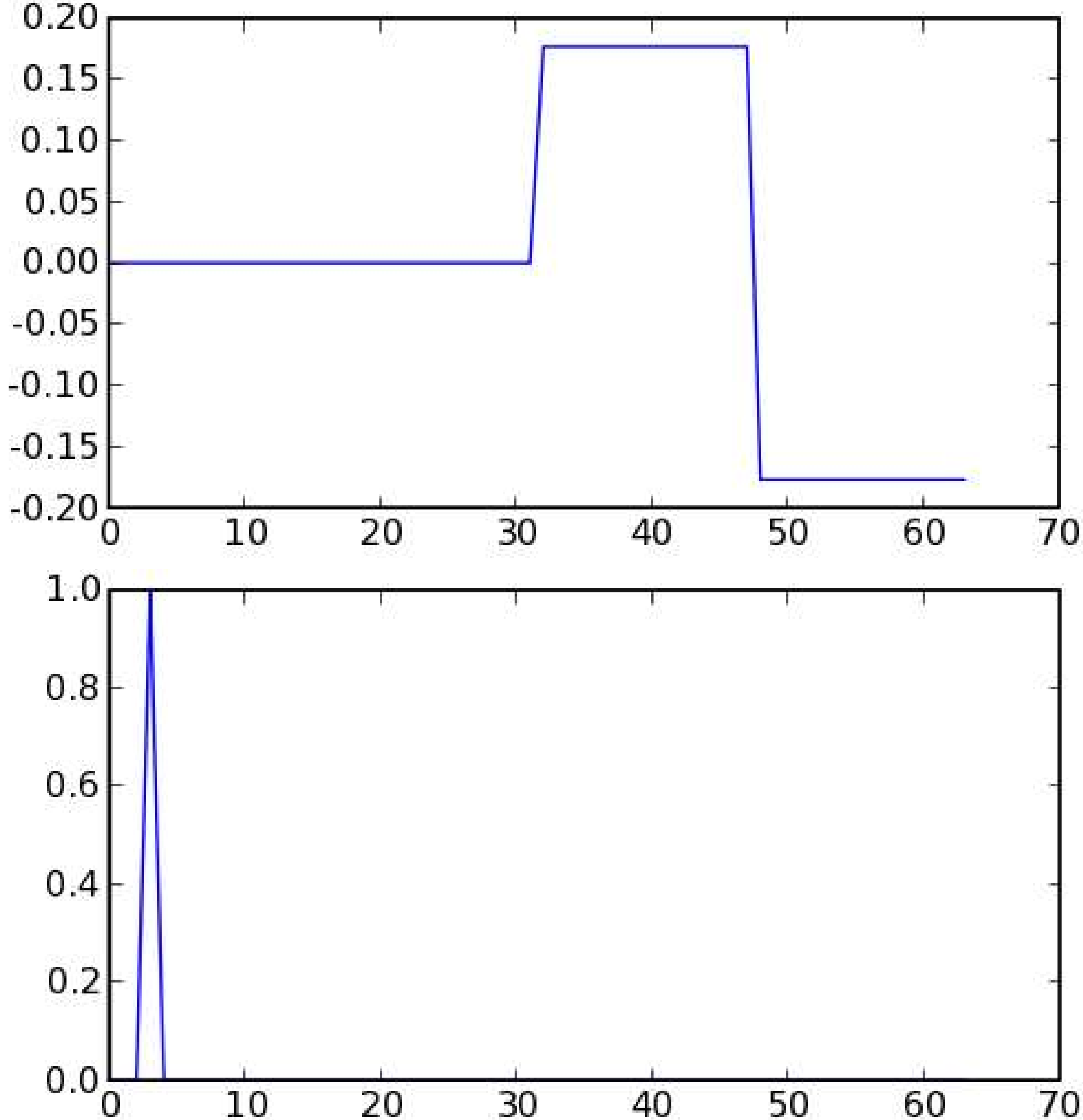
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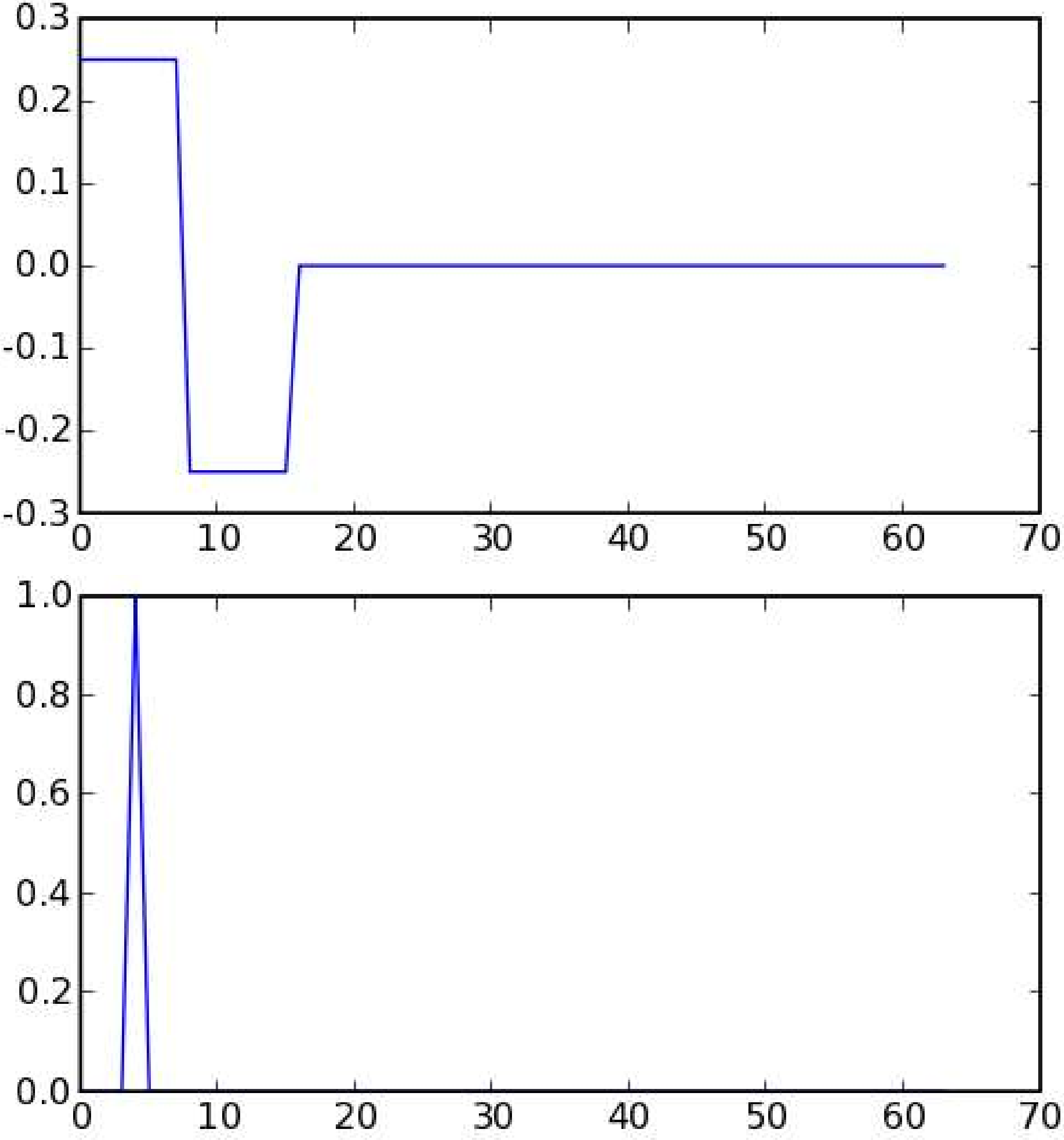
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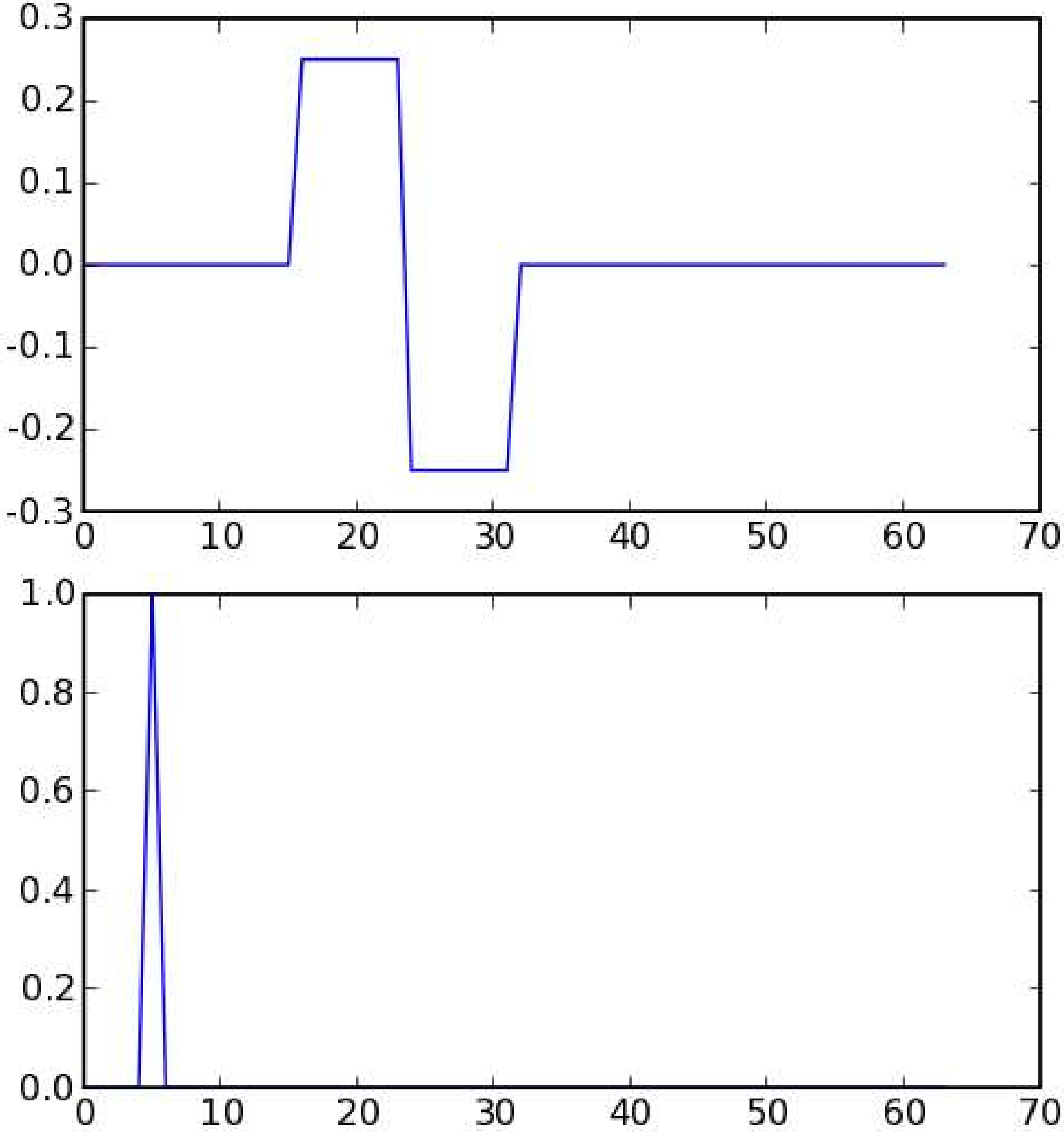
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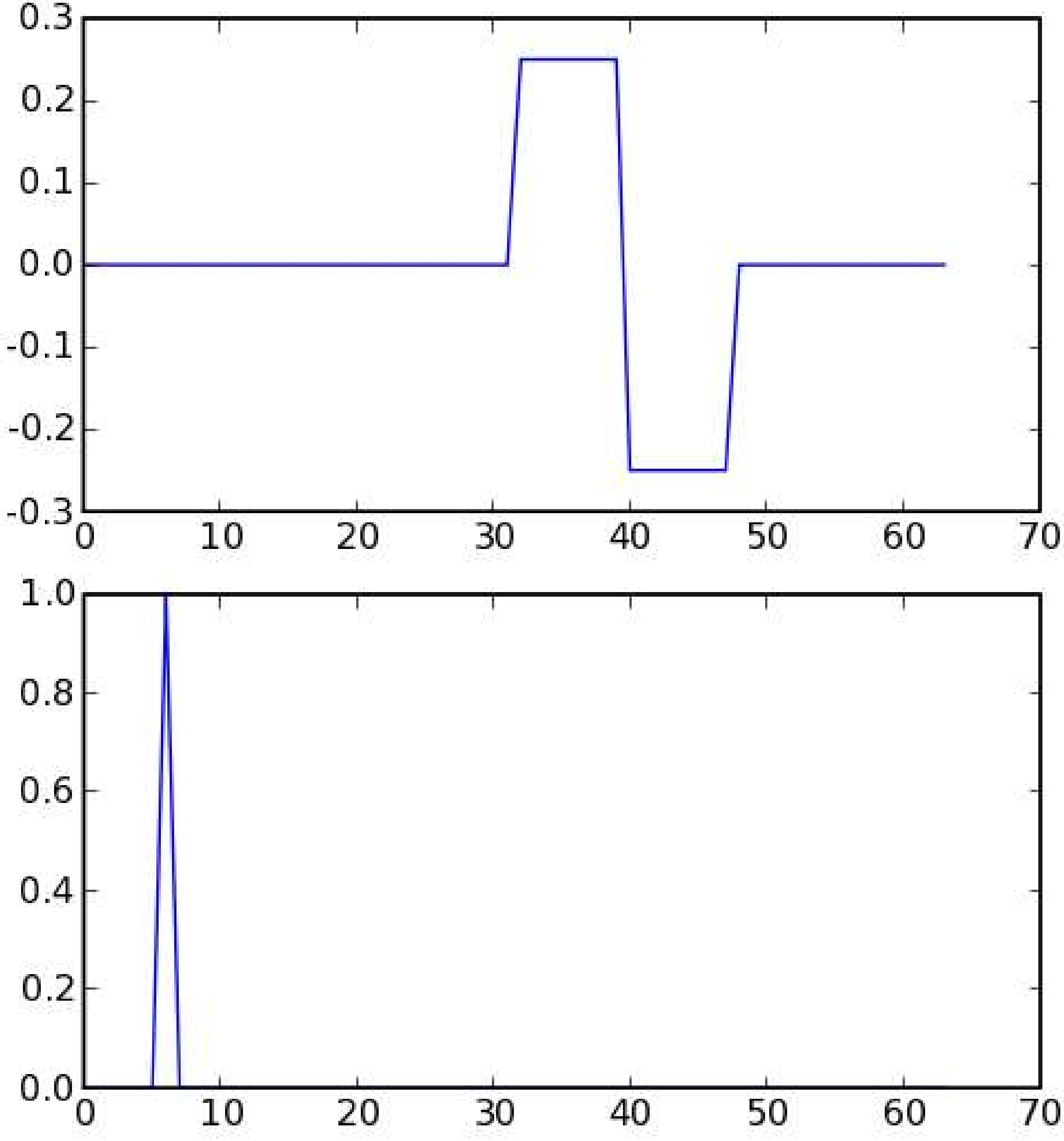

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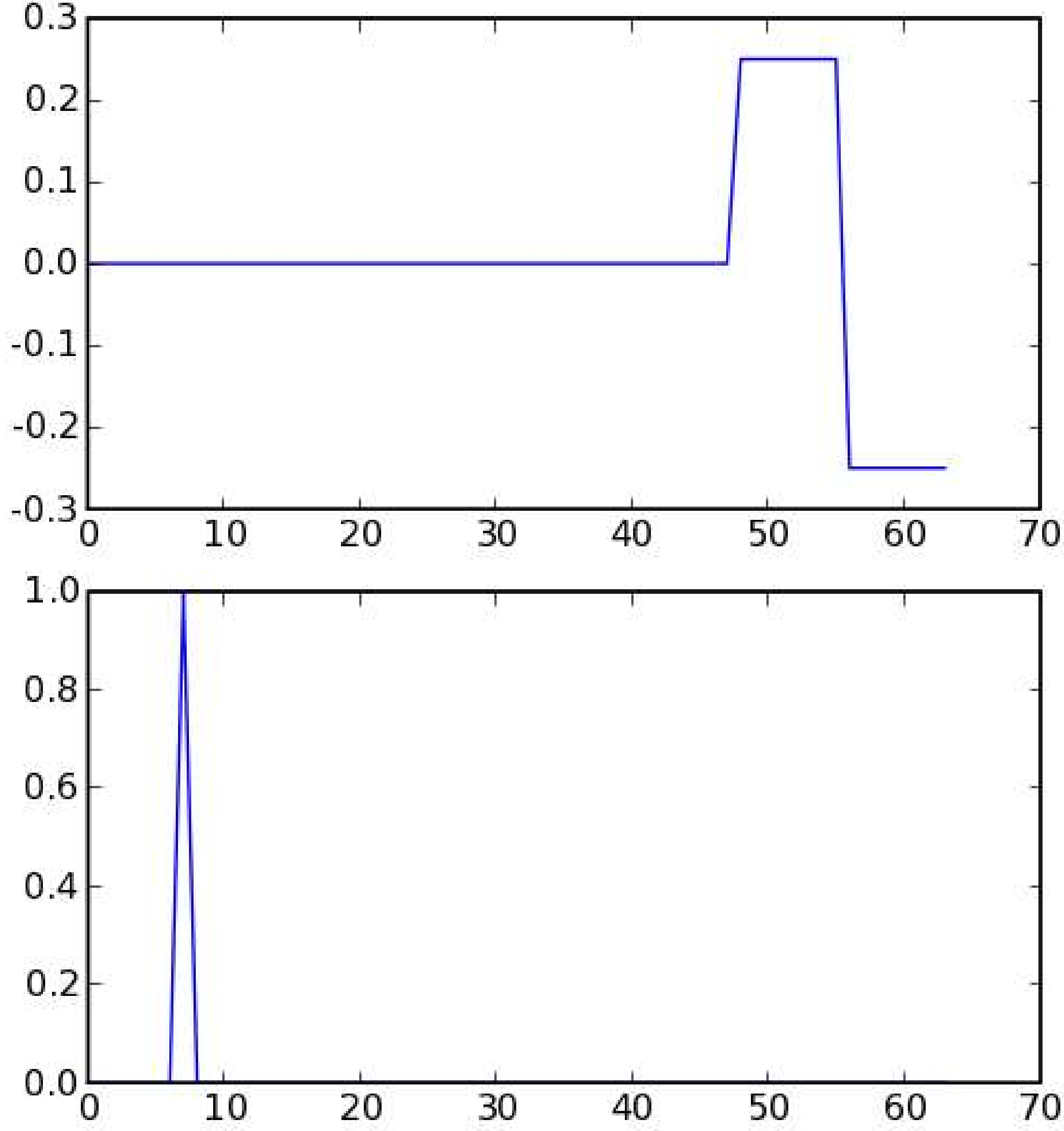
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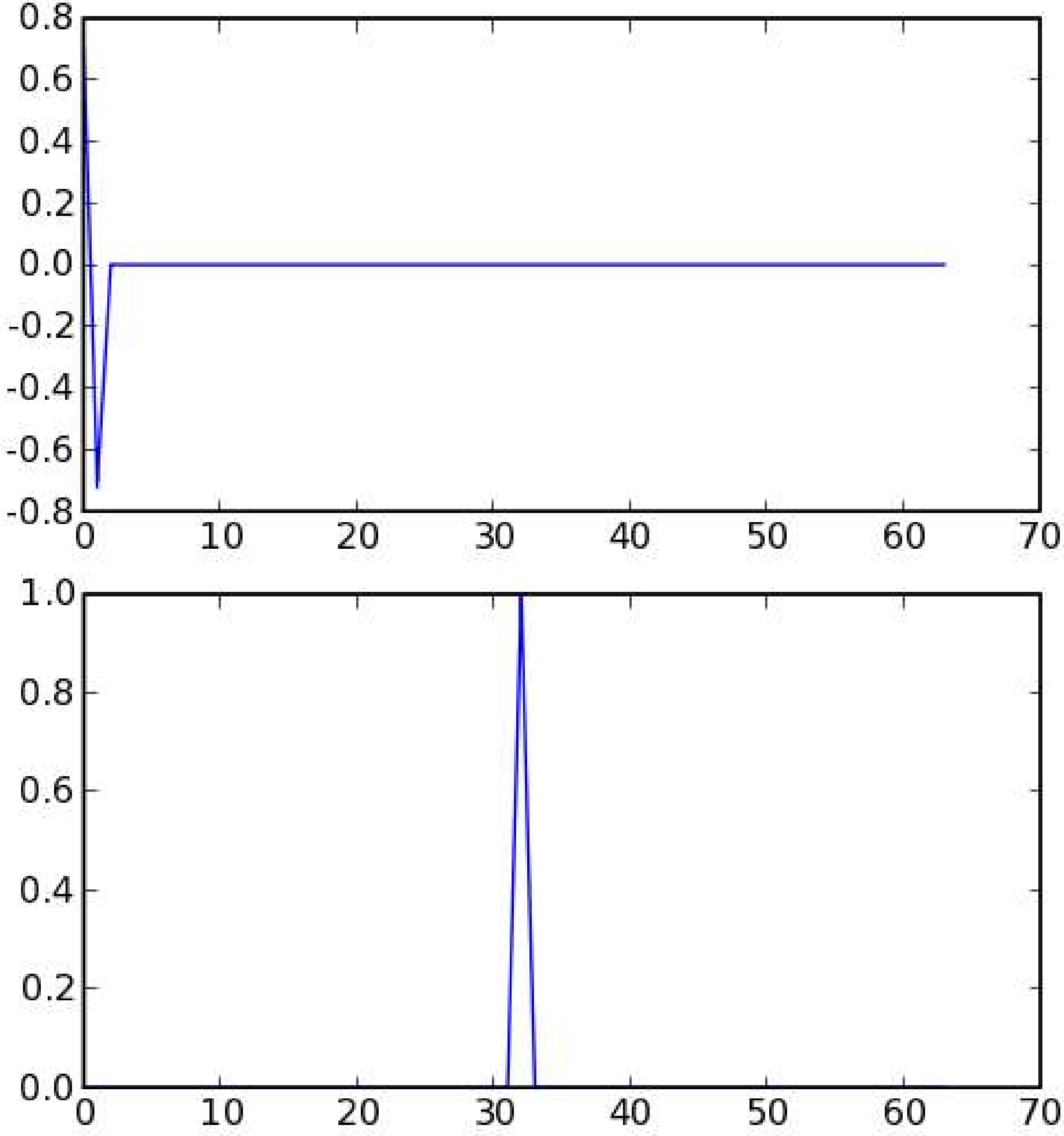

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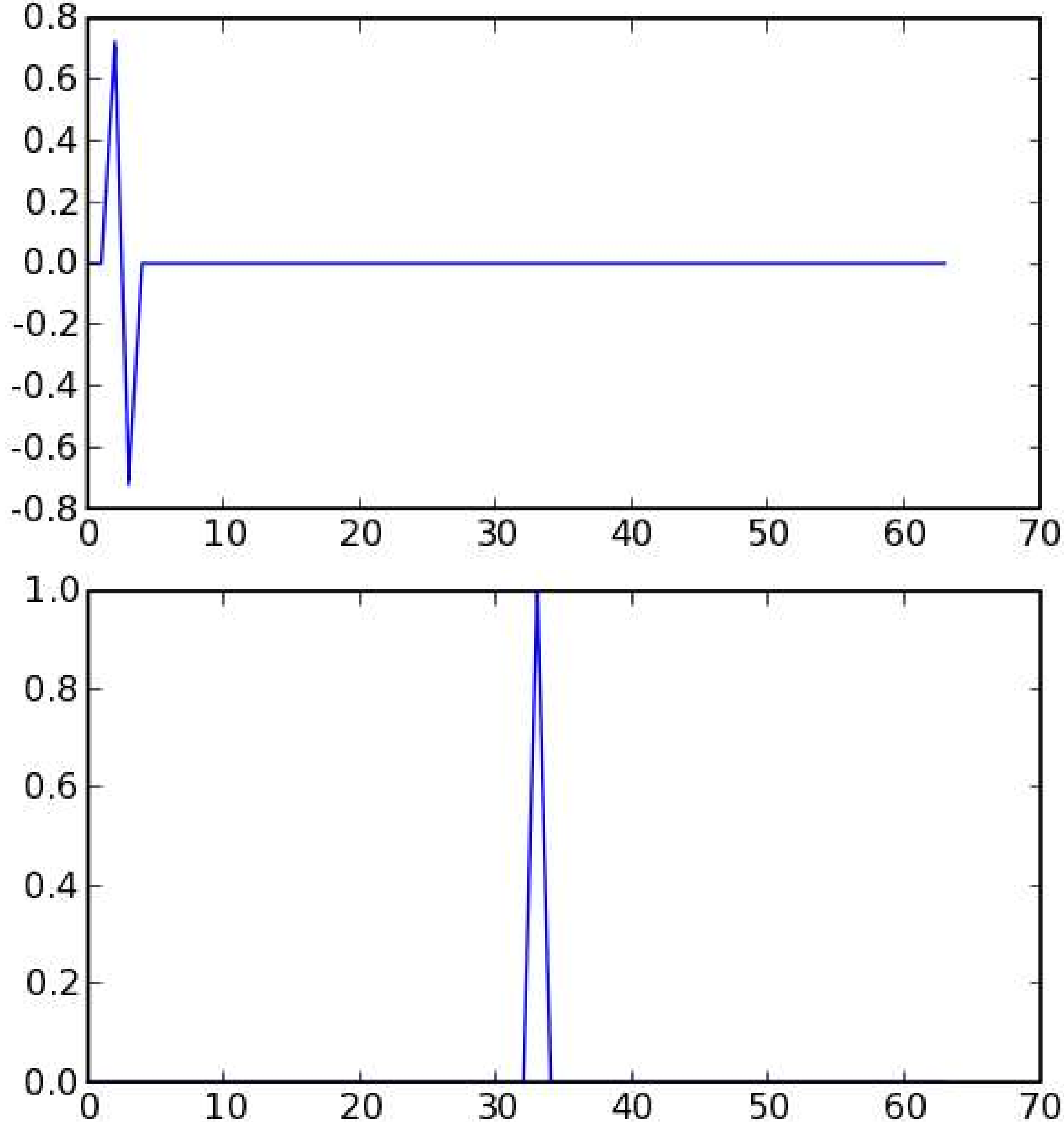
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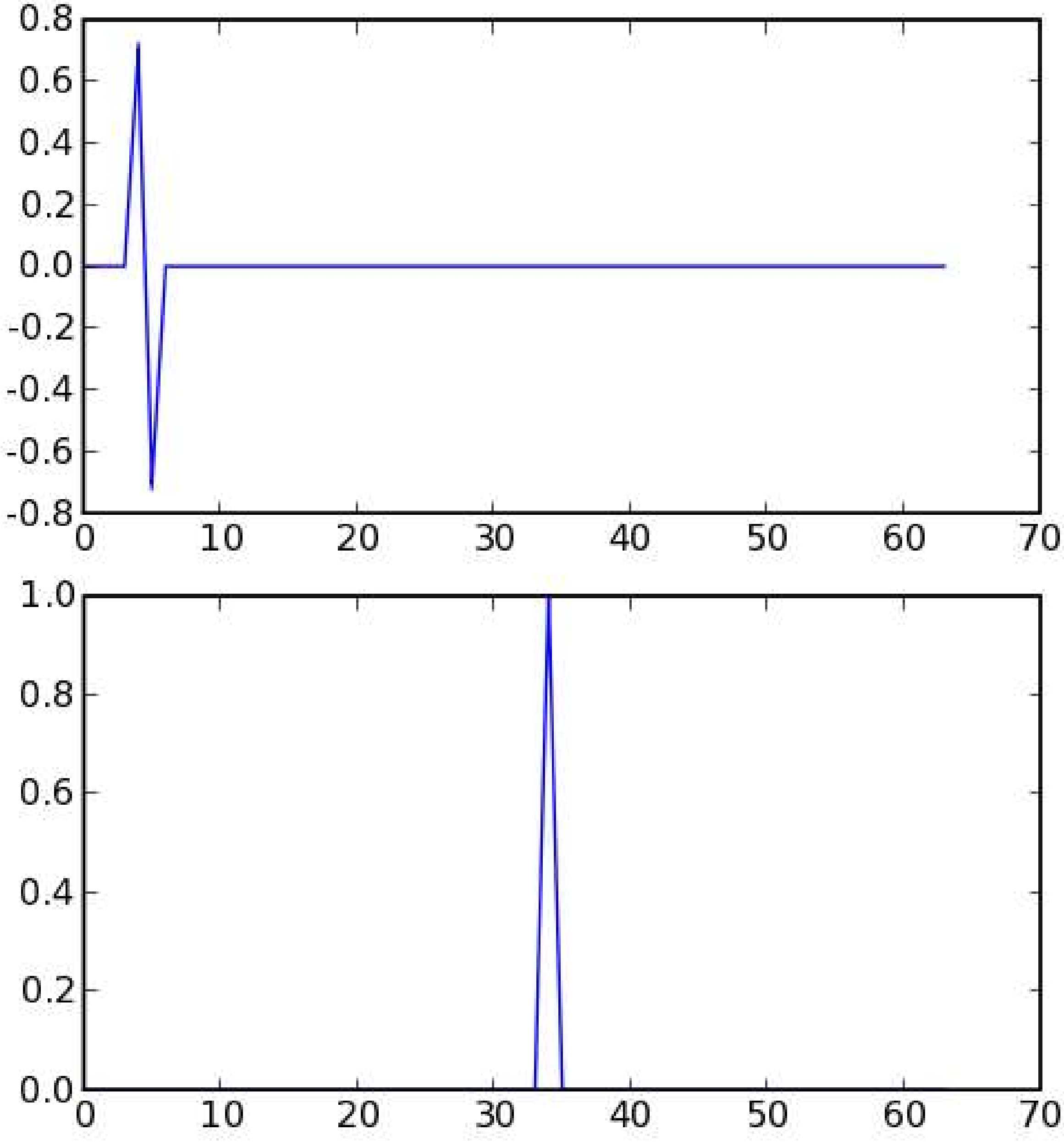
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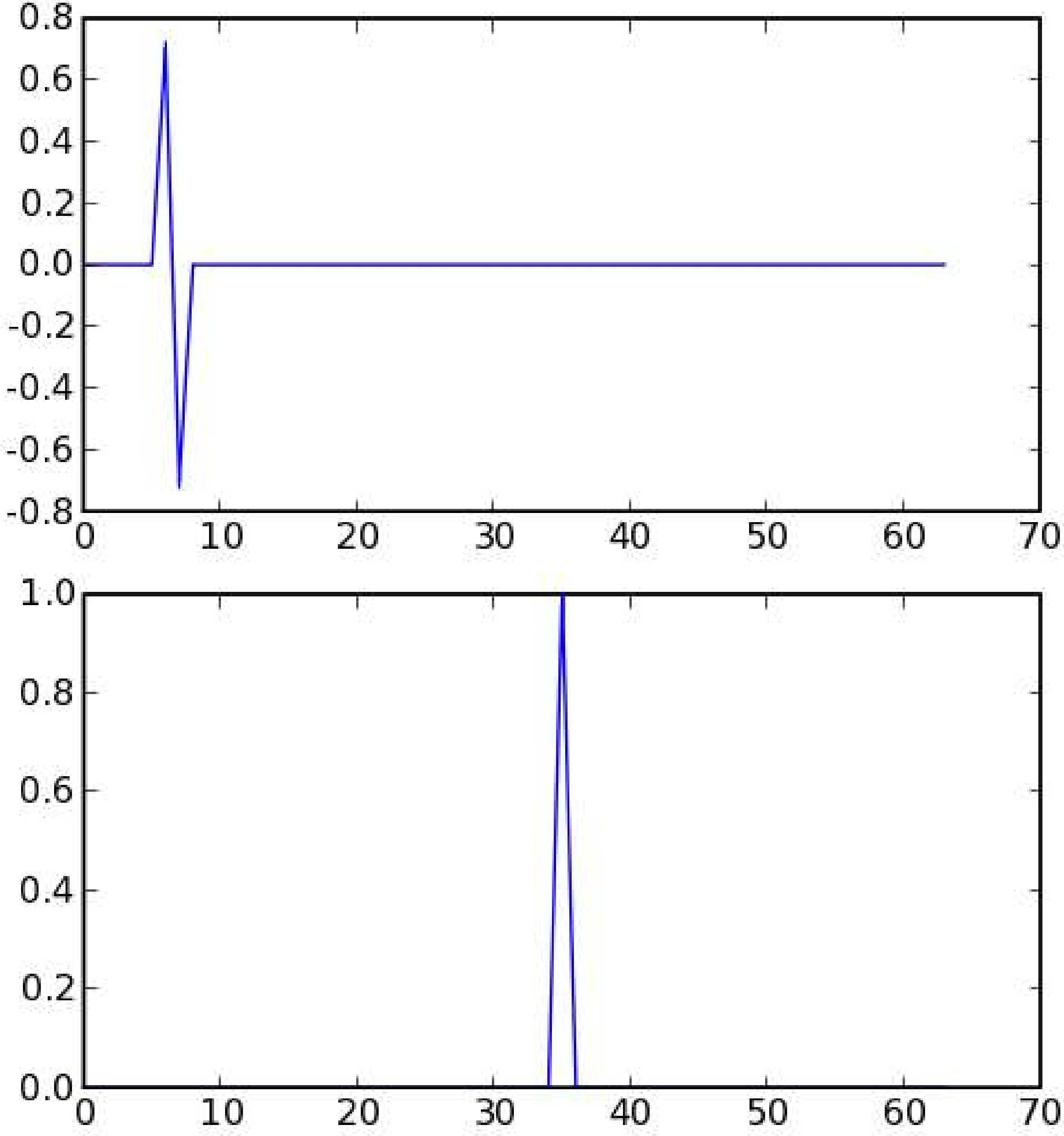

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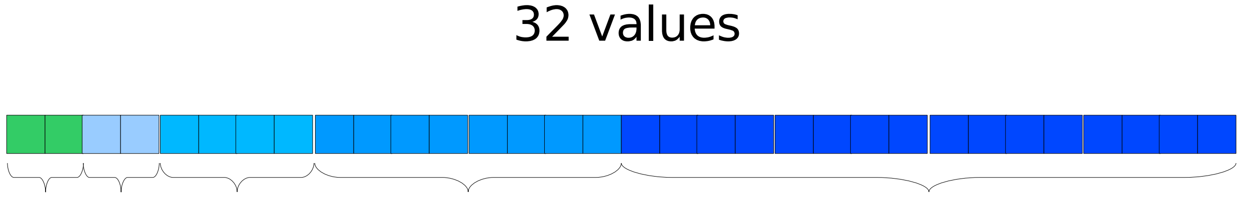
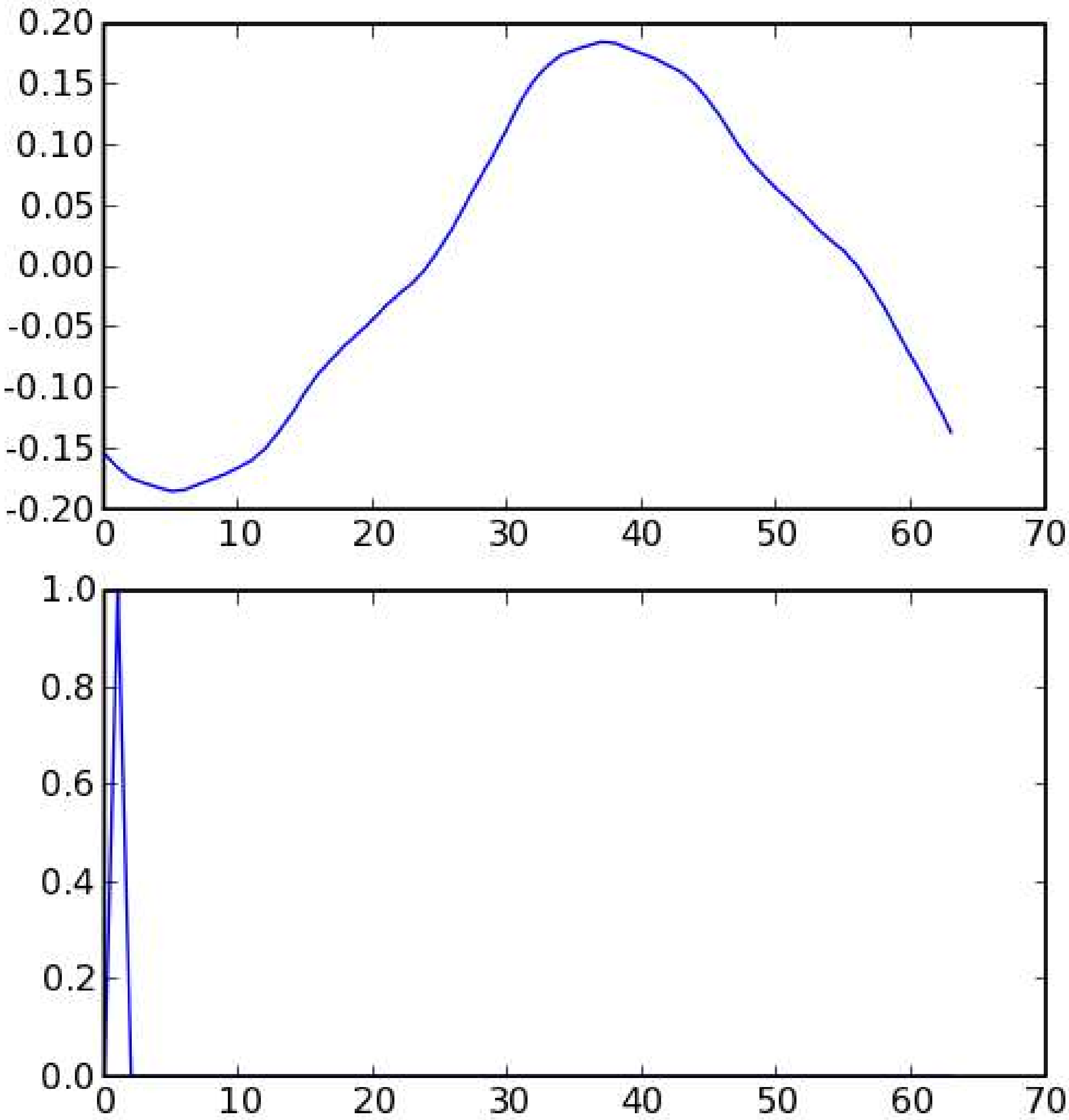
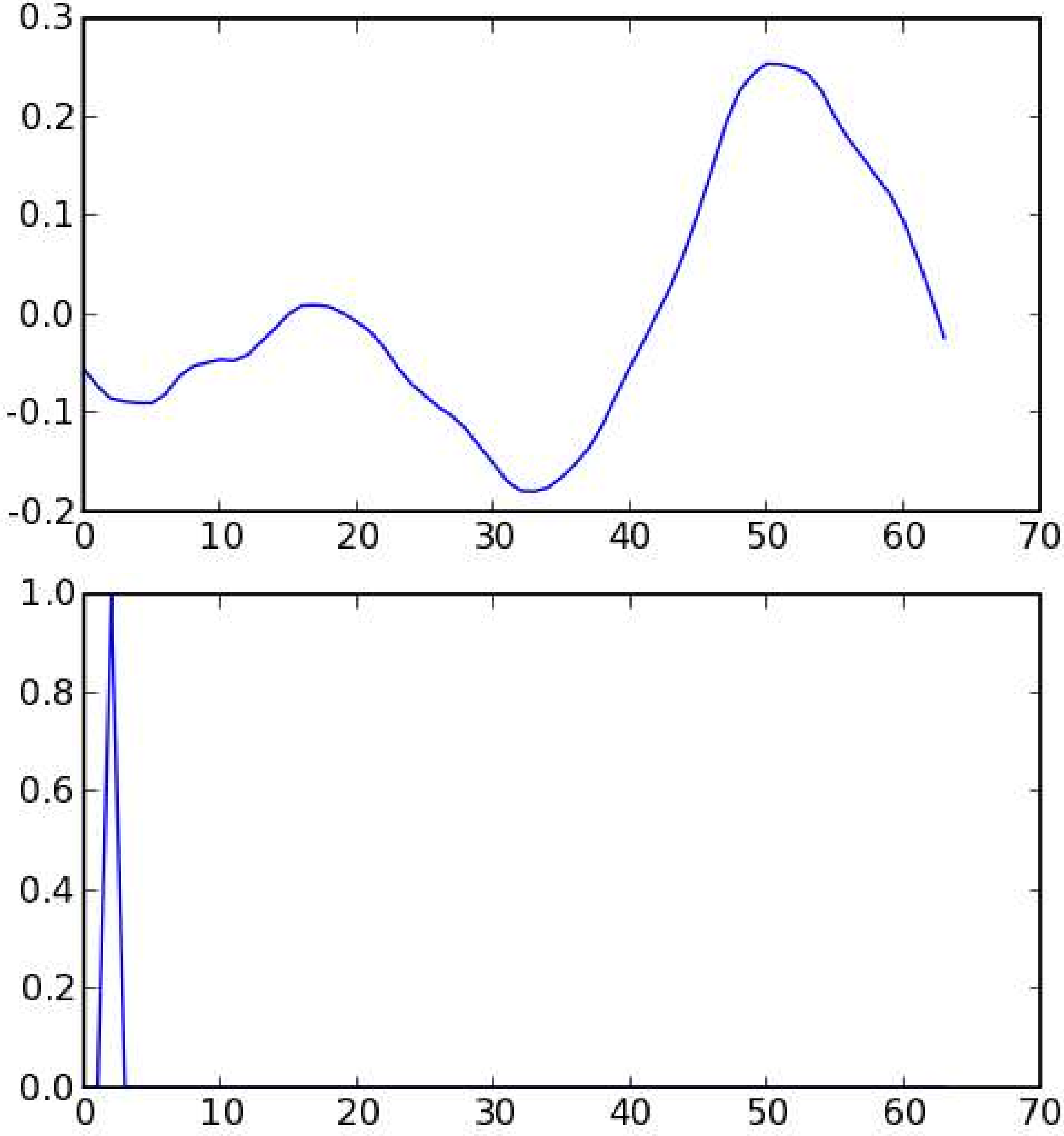
Wavelet Representation



Daubichies Wavelet (6)



Daubichies Wavelet (6)



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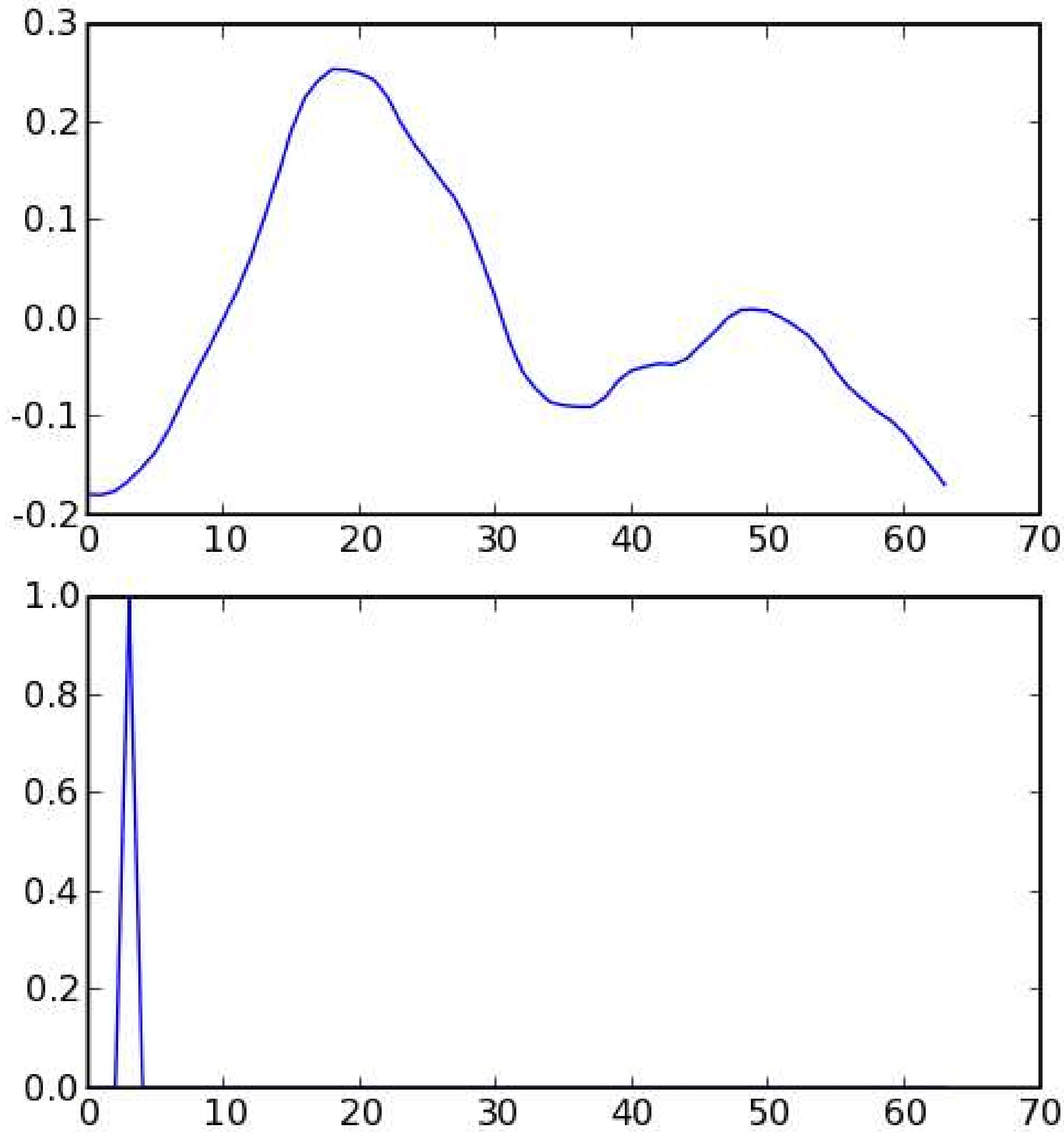

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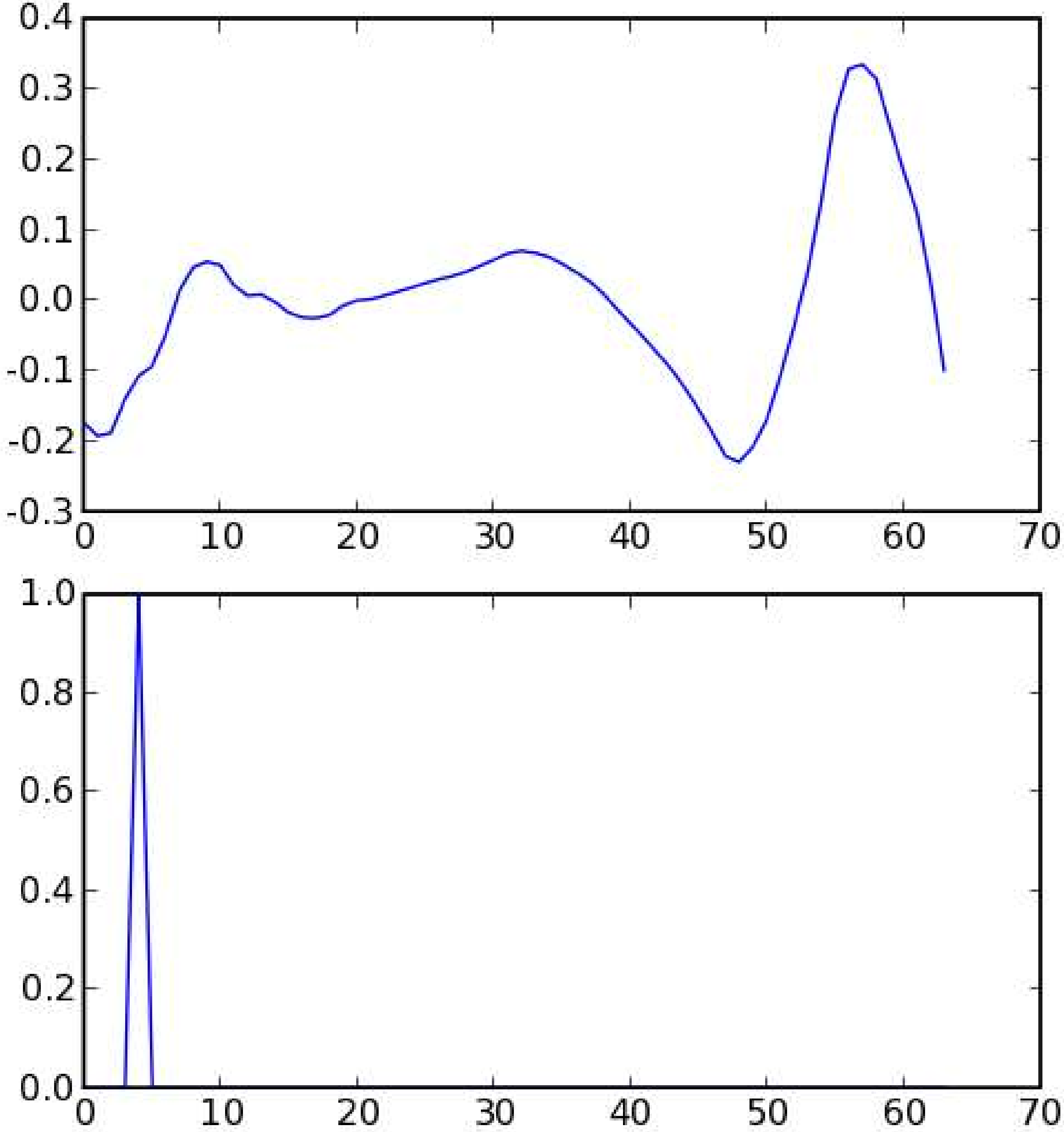

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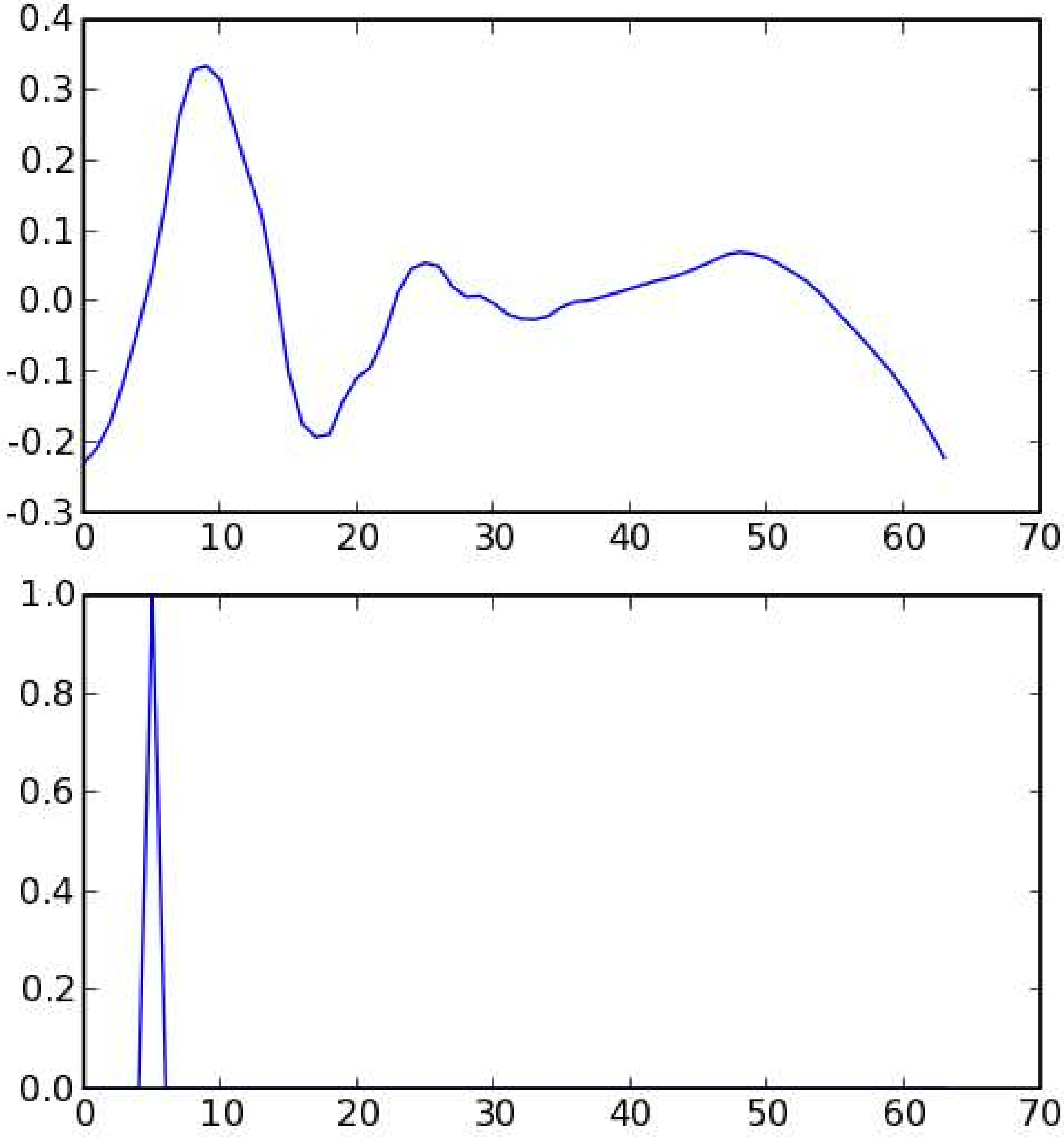

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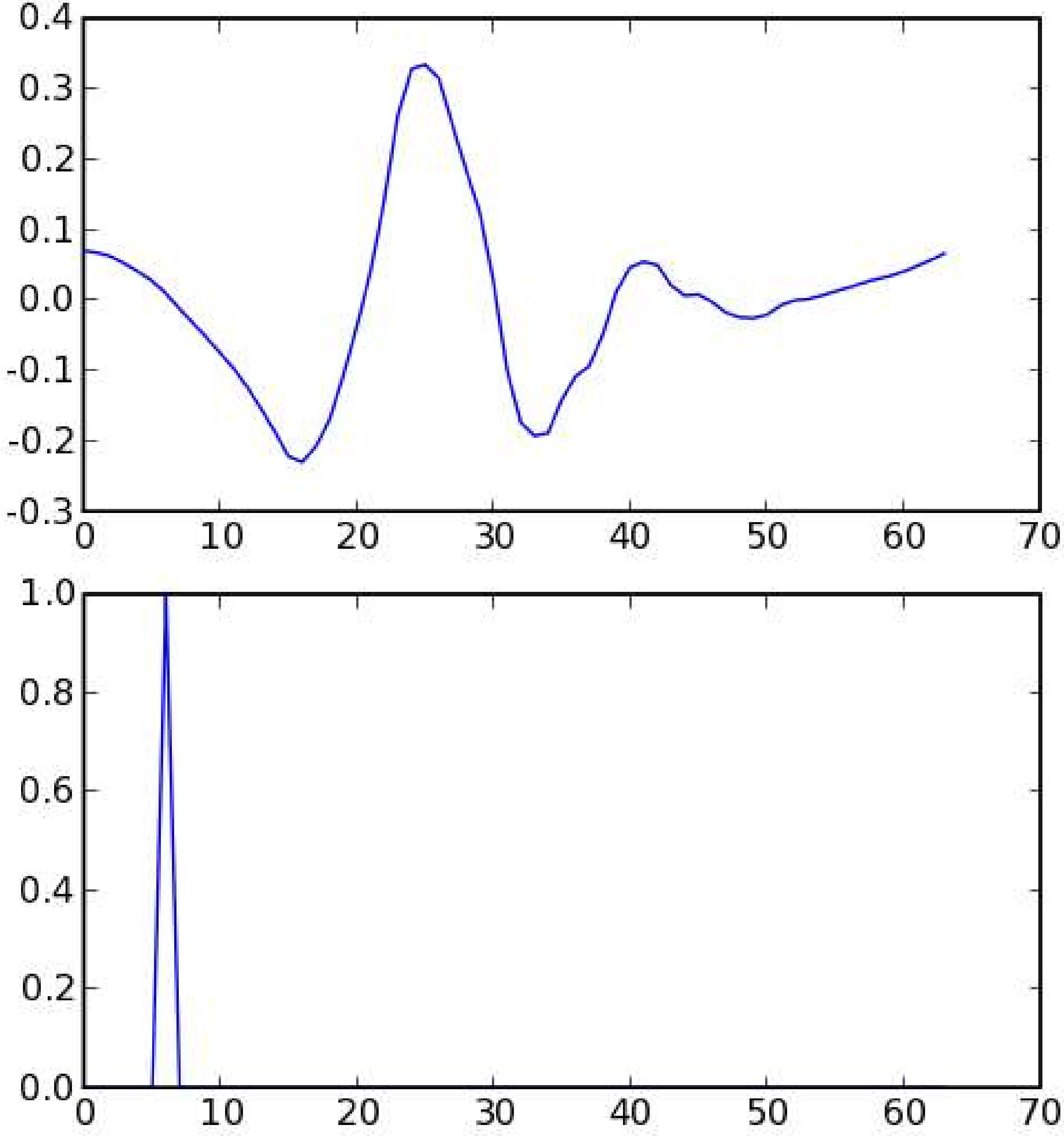

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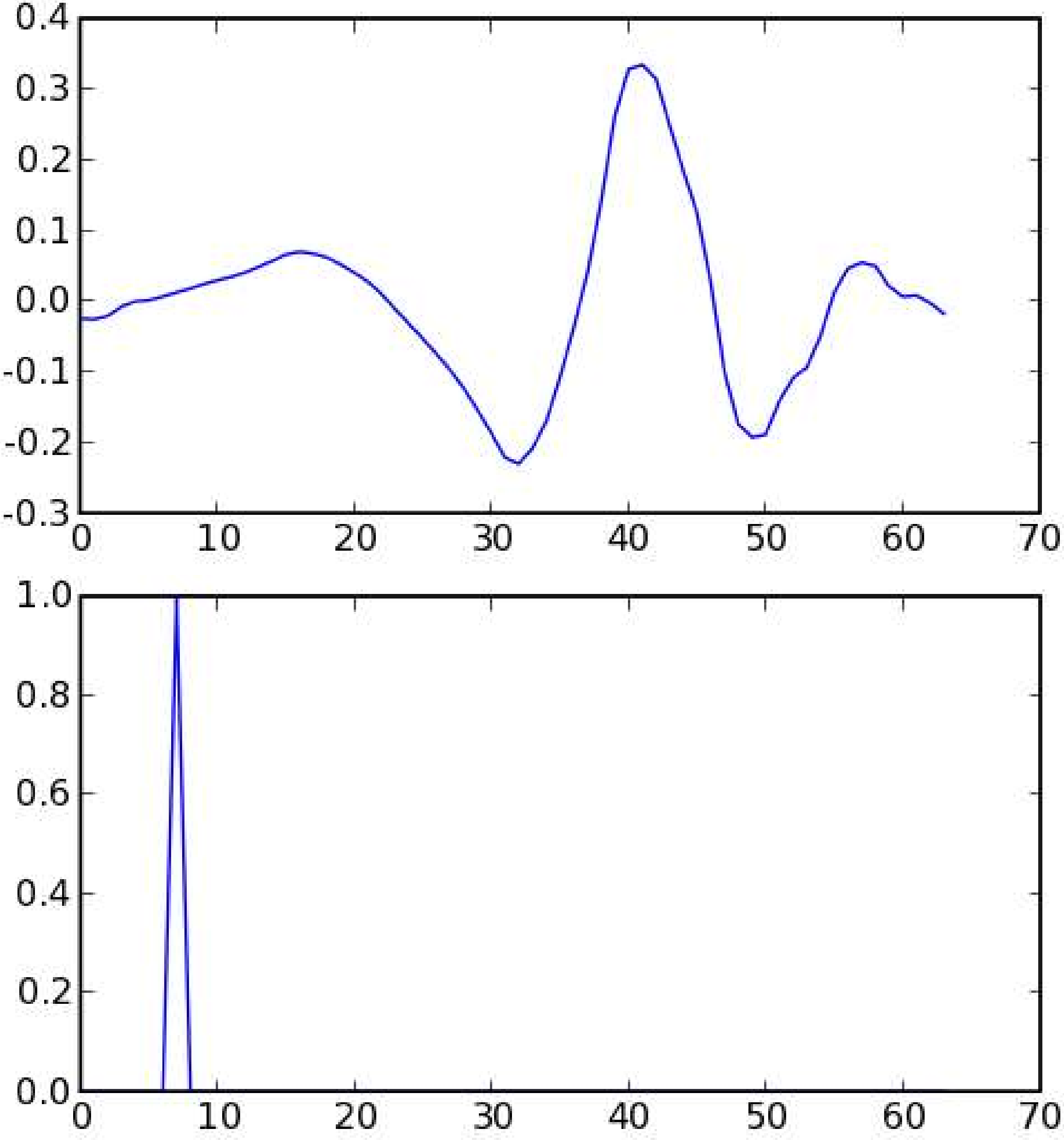
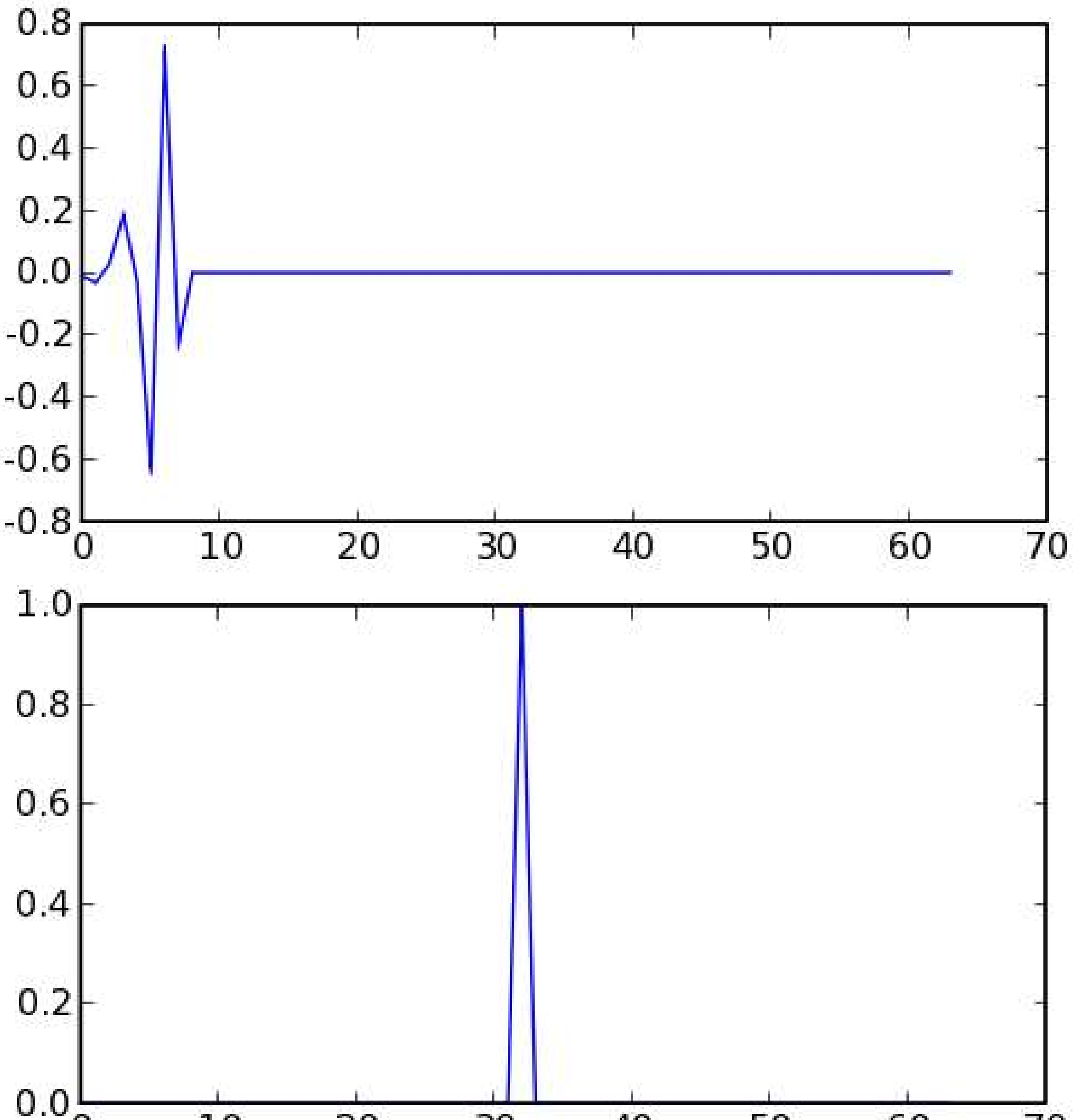


Daubichies Wavelet (6)



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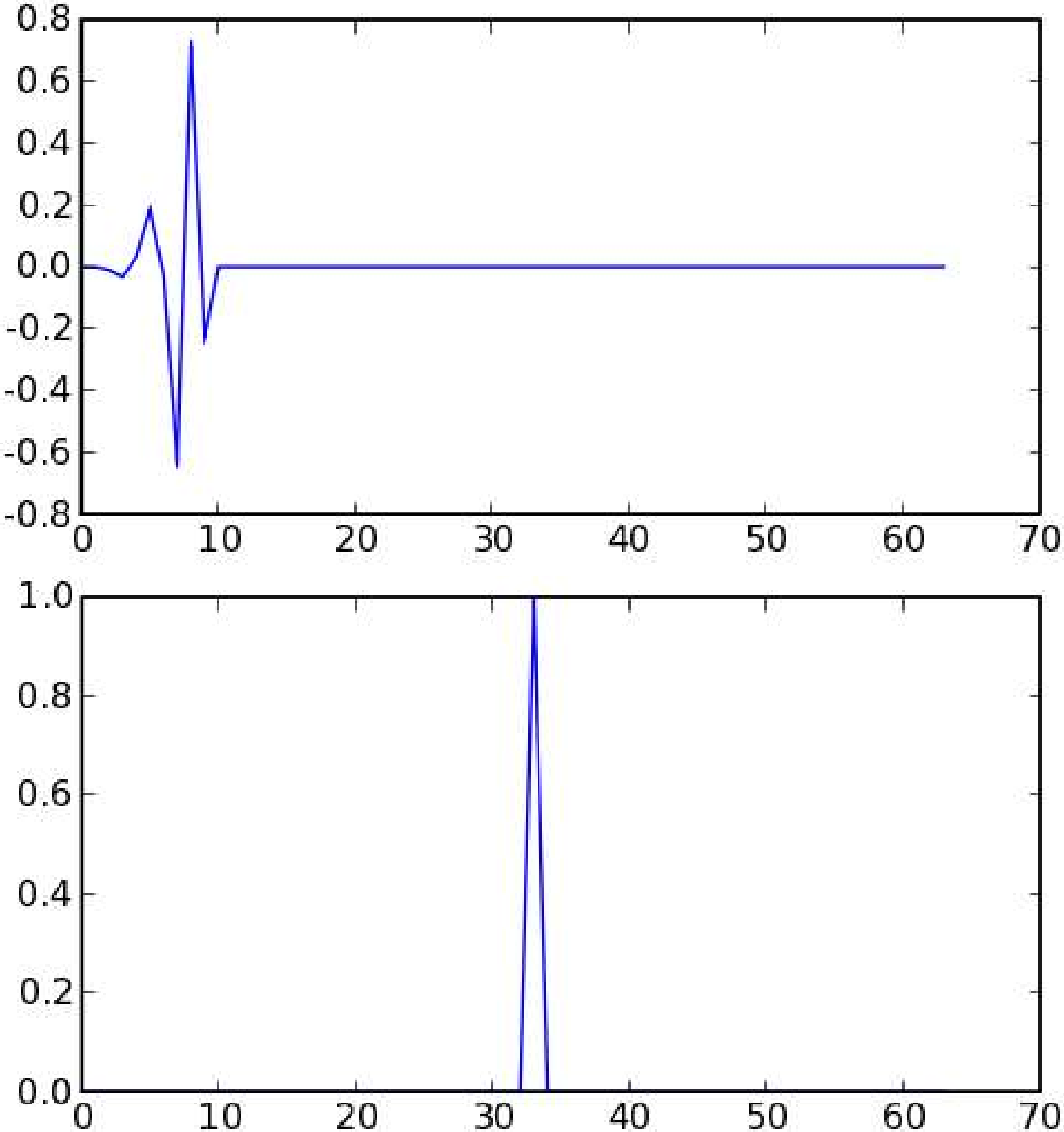

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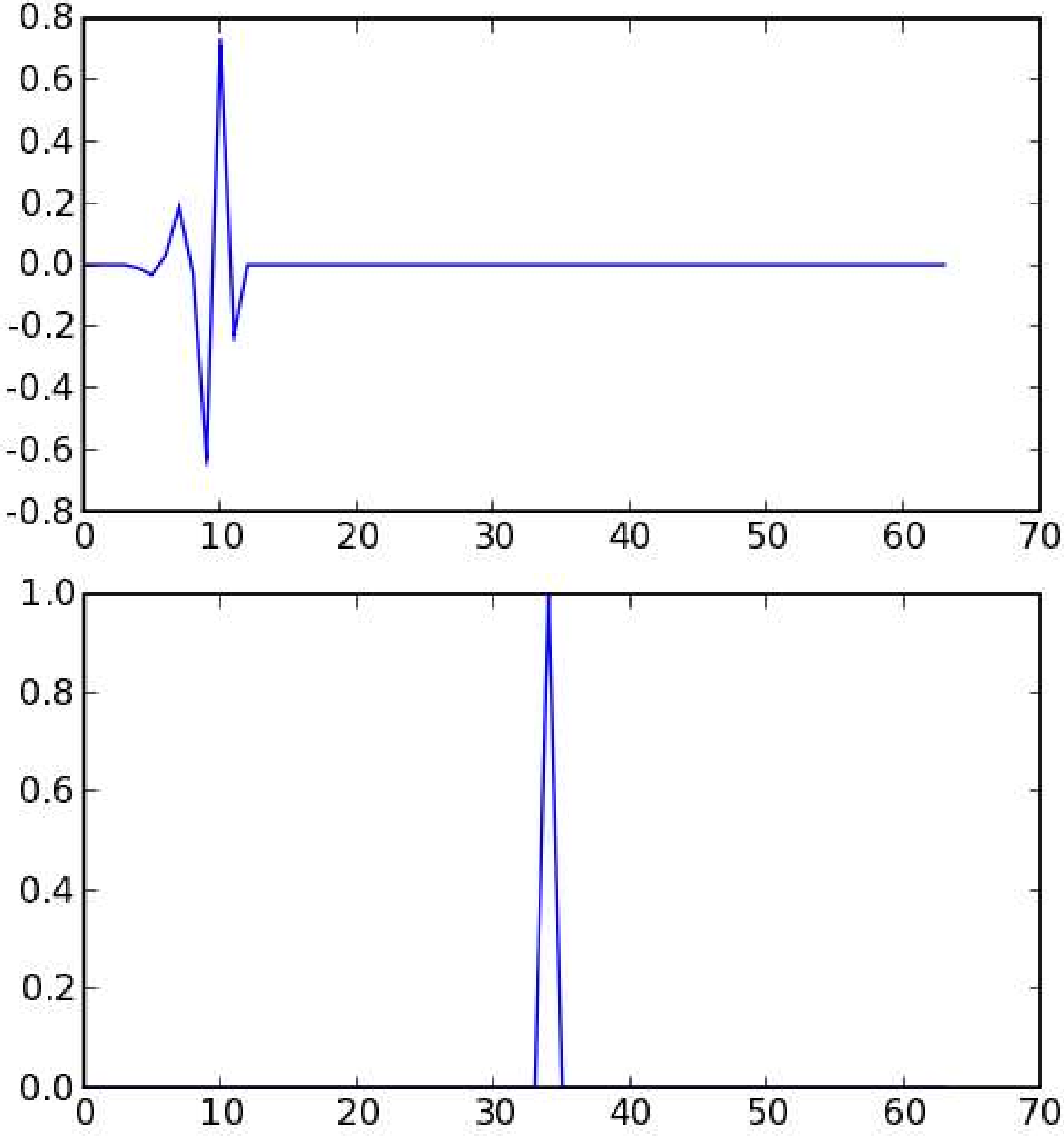

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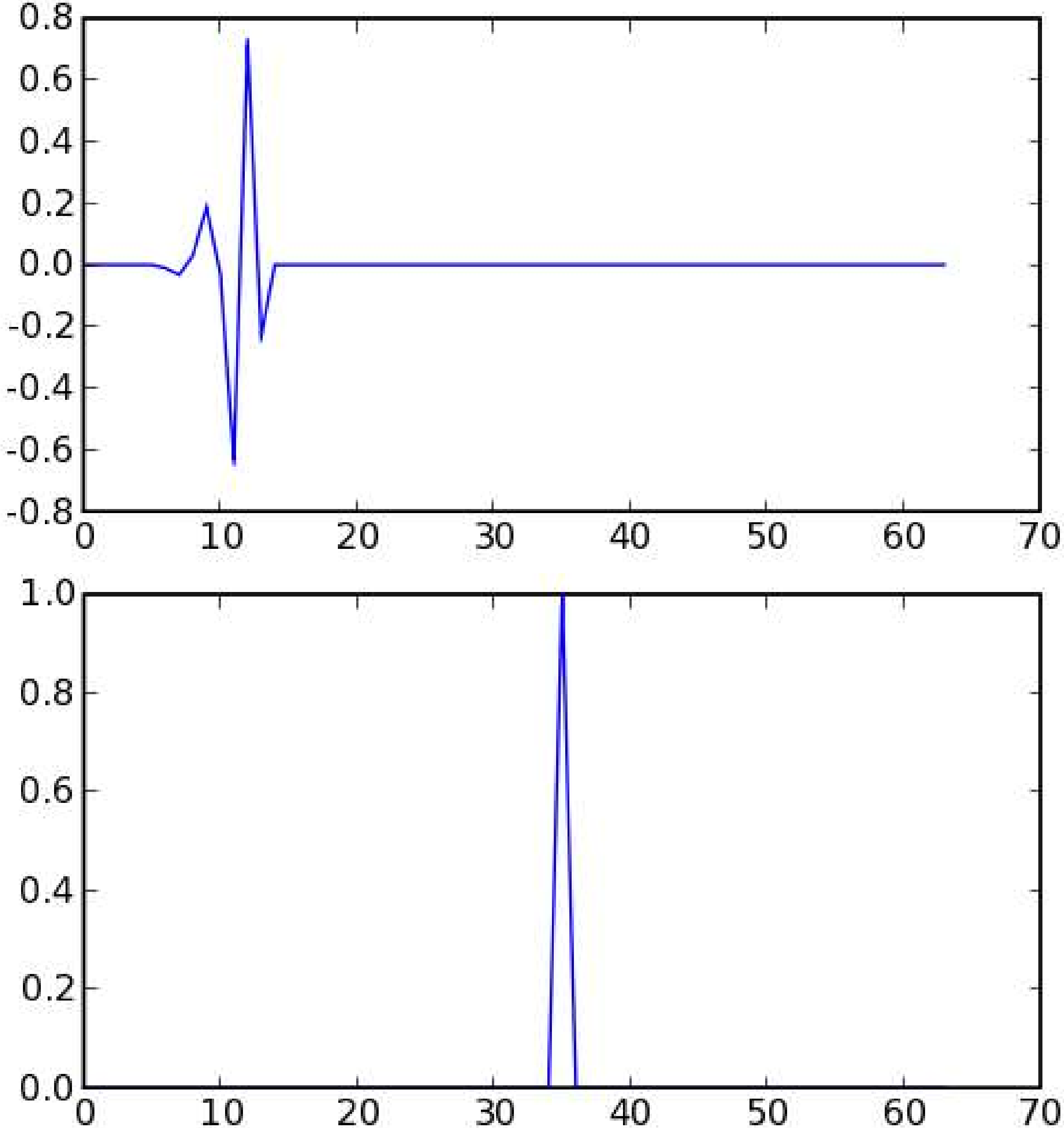
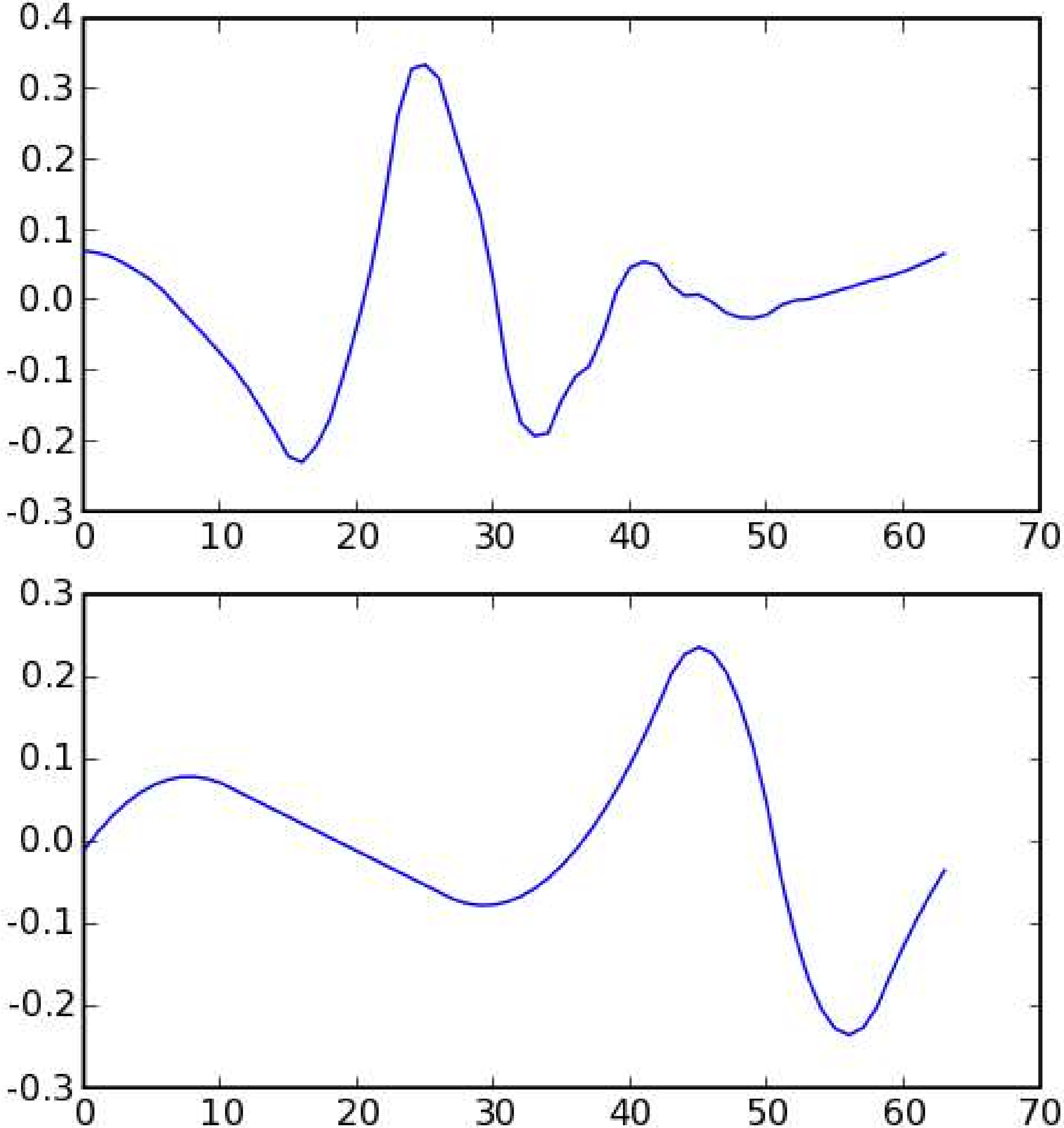
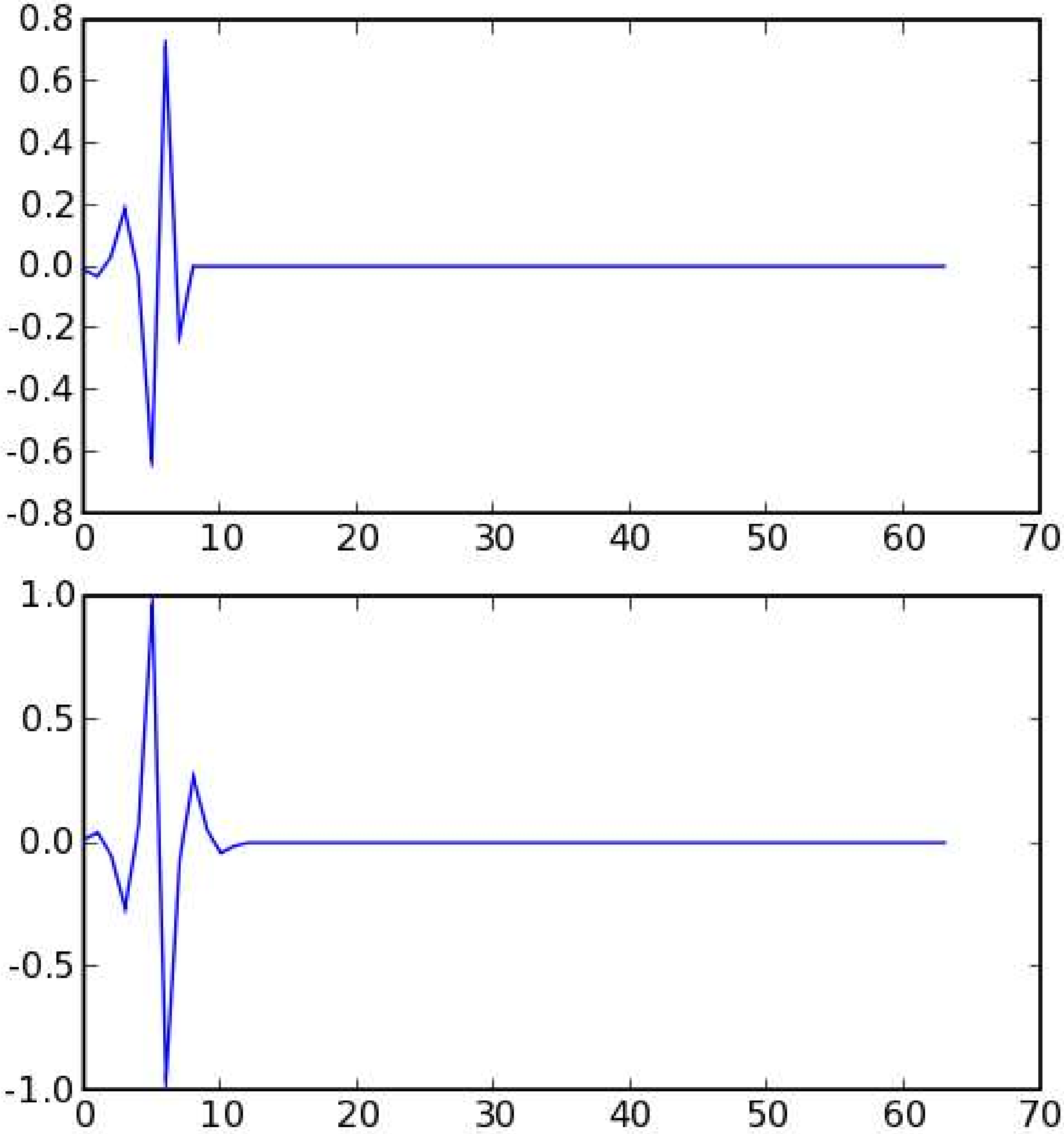


Daubichies(6) vs. B-Spline(304)



Daubichies(6) vs. B-Spline(304)



$$\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-ikx} \rightarrow$$

$$\begin{bmatrix} \bar{F}_0 & \bar{F}_1 & \bar{F}_2 & \bar{F}_3 \end{bmatrix} = \begin{bmatrix} e^{-ik_0x_0} & e^{-ik_0x_1} & e^{-ik_0x_2} & e^{-ik_0x_3} \\ e^{-ik_1x_0} & e^{-ik_1x_1} & e^{-ik_1x_2} & e^{-ik_1x_3} \\ e^{-ik_2x_0} & e^{-ik_2x_1} & e^{-ik_2x_2} & e^{-ik_2x_3} \\ e^{-ik_3x_0} & e^{-ik_3x_1} & e^{-ik_3x_2} & e^{-ik_3x_3} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Orthogonality

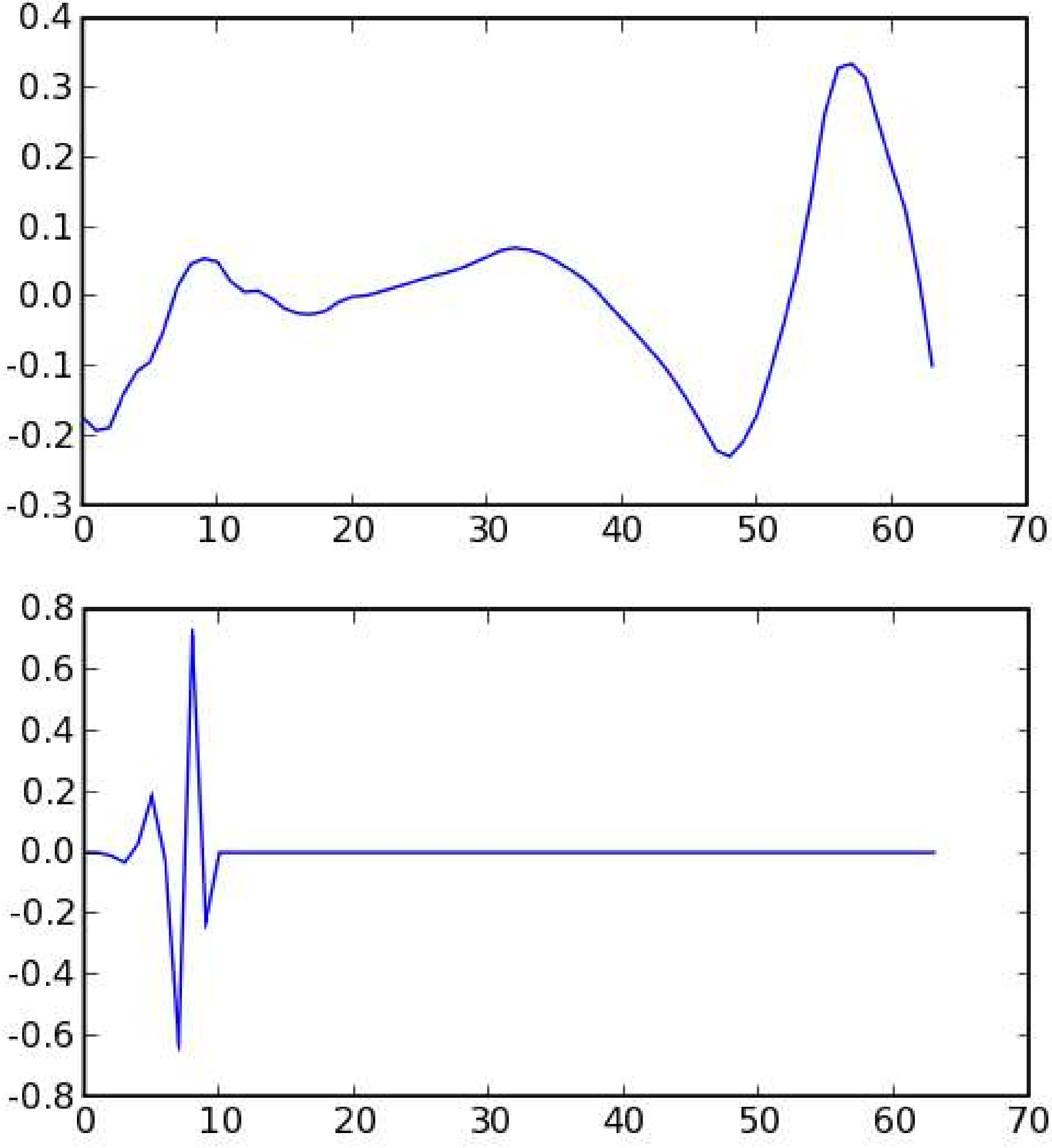
$$\int_{-\infty}^{\infty} \sin(k_1 x)\sin(k_2 x)\,dx = 0 \qquad (k_1 \neq k_2)$$

Orthogonality

$$\int_{-\infty}^{\infty} \sin(k_1 x)\sin(k_2 x)\,dx = 0 \qquad (k_1 \neq k_2)$$

$$\sum_{x=0}^{w} \sin\left(2\pi \frac{k_1 x}{w}\right)\sin\left(2\pi \frac{k_2 x}{w}\right) dx = 0 \qquad (k \text{ integer}, k_1 \neq k_2)$$

Orthogonality



SVD / MSA ?

- Real Space –
  - accurate position
  - no spectral information
- FFT
  - no positional information
  - accurate spectral information
- Wavelets
  - mix of power and position
- MSA/SVD
  - data-based basis

Computational Efficiency

Fourier (Matrix Method) : $O(n^2)$
Fourier (FFT) : $O(n \log n)$
Wavelet (std) : $O(n)$

Image Processing

- Filtration
- Deconvolution
- Transformation
- Registration
- Measures of Similarity
- Projection/Reconstruction (3D ->2D, 2D->3D)
- Segmentation

- Dimensionality ?

Fourier Transform Theorems

$$\text{if } f(x)\,\text{real} \rightarrow \bar{F}(k) = \bar{F}^*(-k)$$

$$Convolution: f * g \rightarrow \bar{F}(k)\,\bar{G}(k)$$

$$Correlation: \int_{-\infty}^{\infty} g(x+a)\,h(a)\,da \rightarrow \bar{G}(k)\,\bar{H}^*(k)$$

$$Translation: f(x+x_0) \rightarrow \bar{F}(k)\,e^{i k x_0}$$

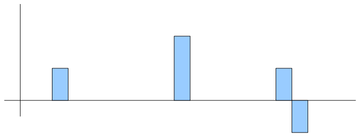
Filtration -> Convolution

Continuous Real Space Convolution:

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(t)g(x-t)dt$$

Discrete Real Space Convolution:

$$f_i*g_i=\sum_{t=-\infty}^{\infty} f_t g_{i-t}$$



Filtration -> Convolution

Continuous Real Space Convolution:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

Discrete Real Space Convolution:

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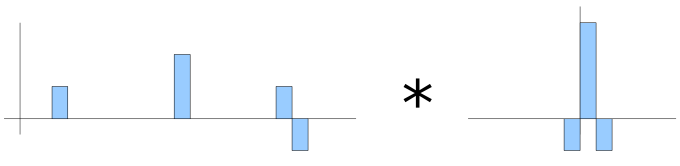

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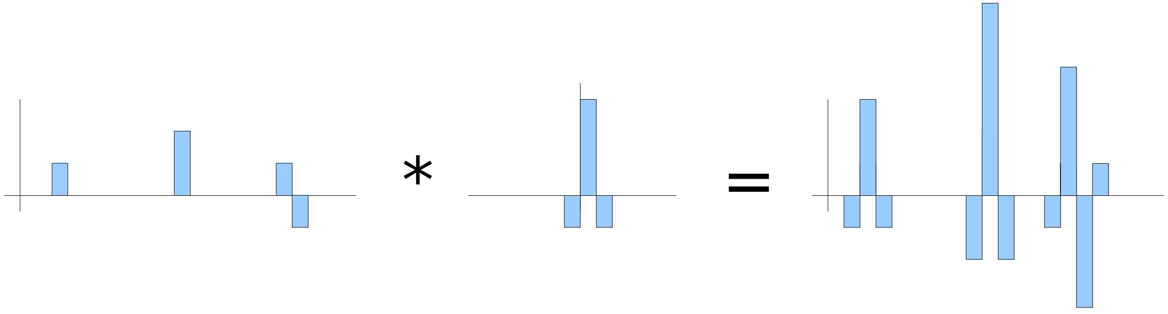
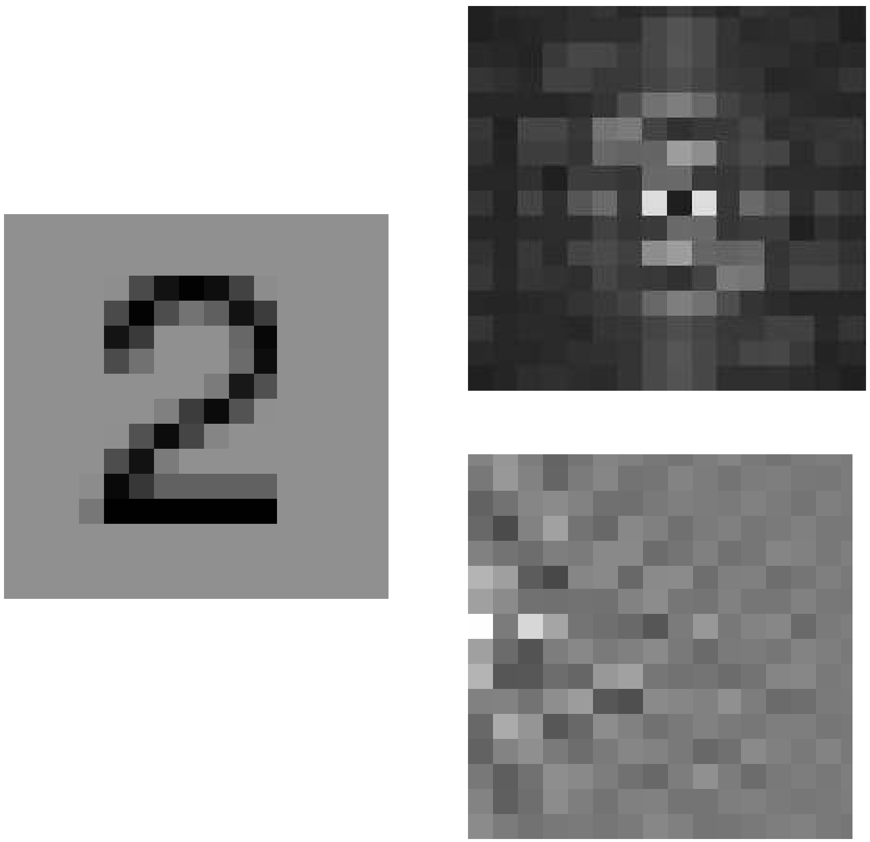
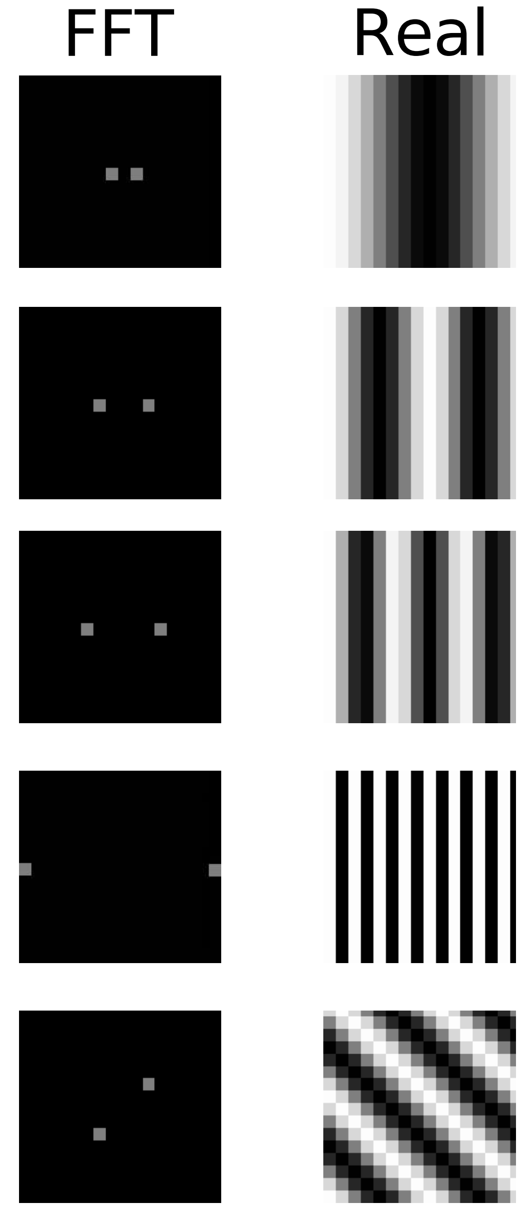


Fourier Space and Images

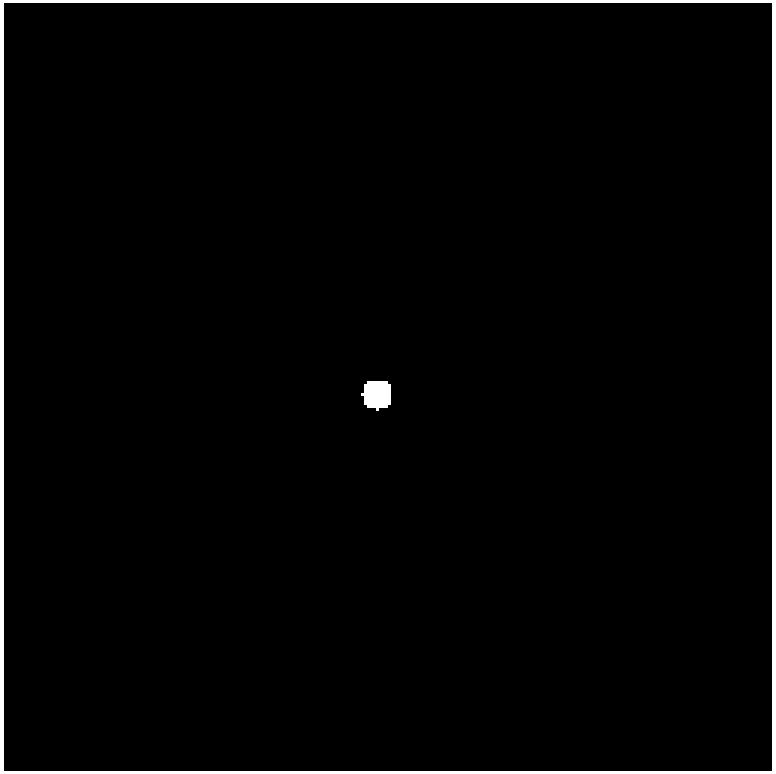


$$F(k)=\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

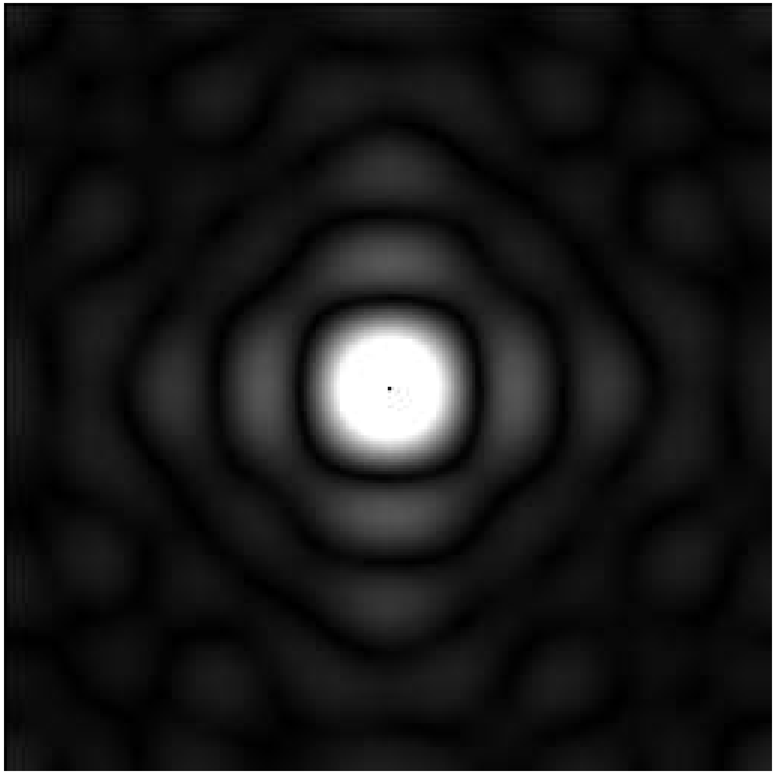
a+ib = amp(cos pha+i sin pha)



FFT Image demo

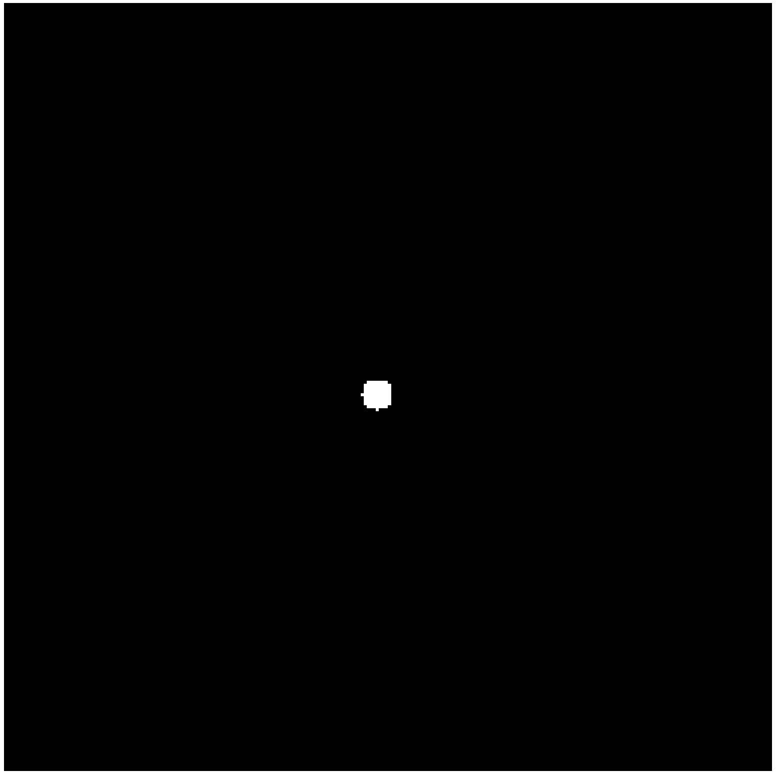


Real

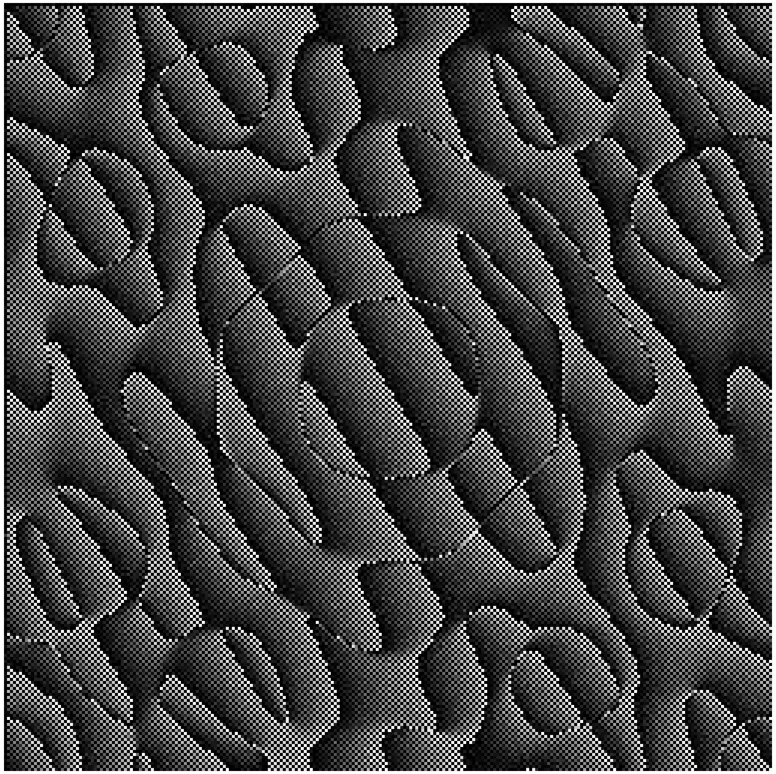


FFT Amplitude

FFT Image demo

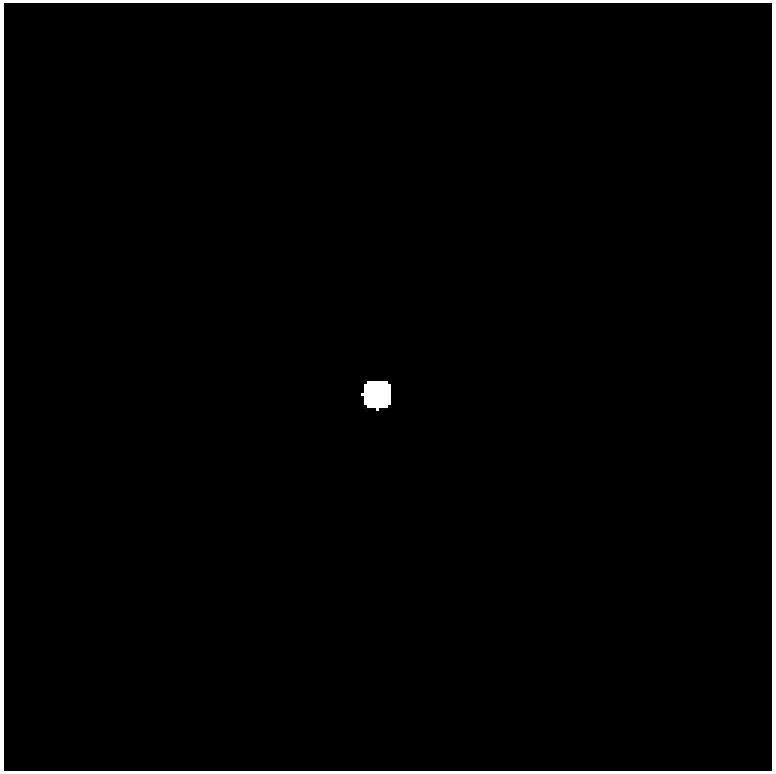


Real

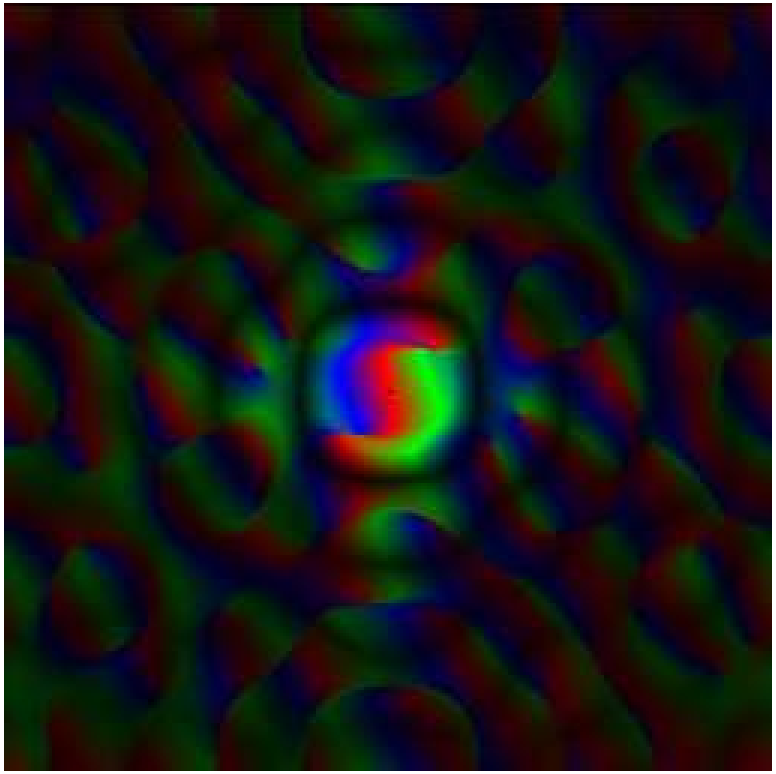


FFT Phase

FFT Image demo

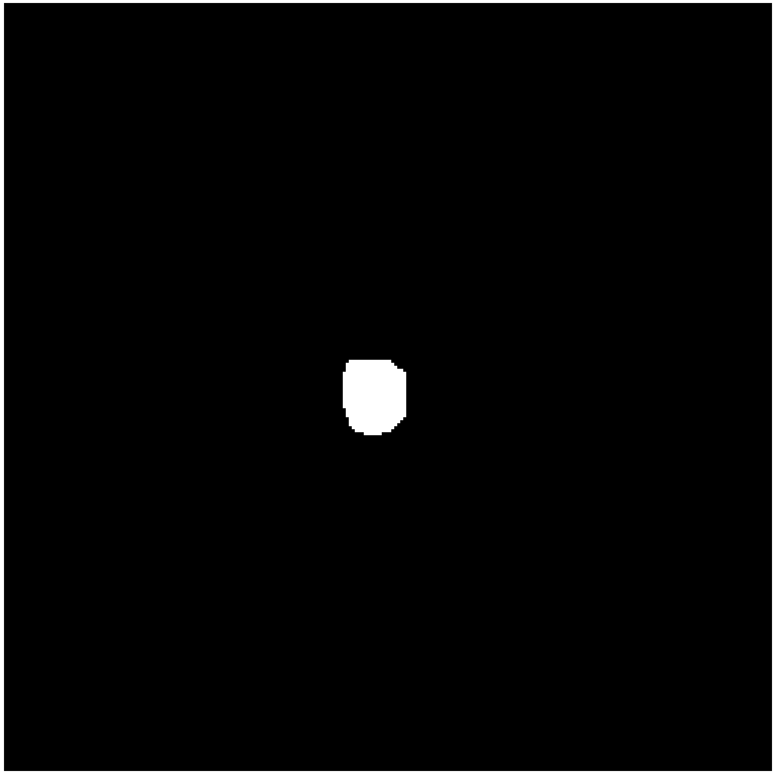


Real

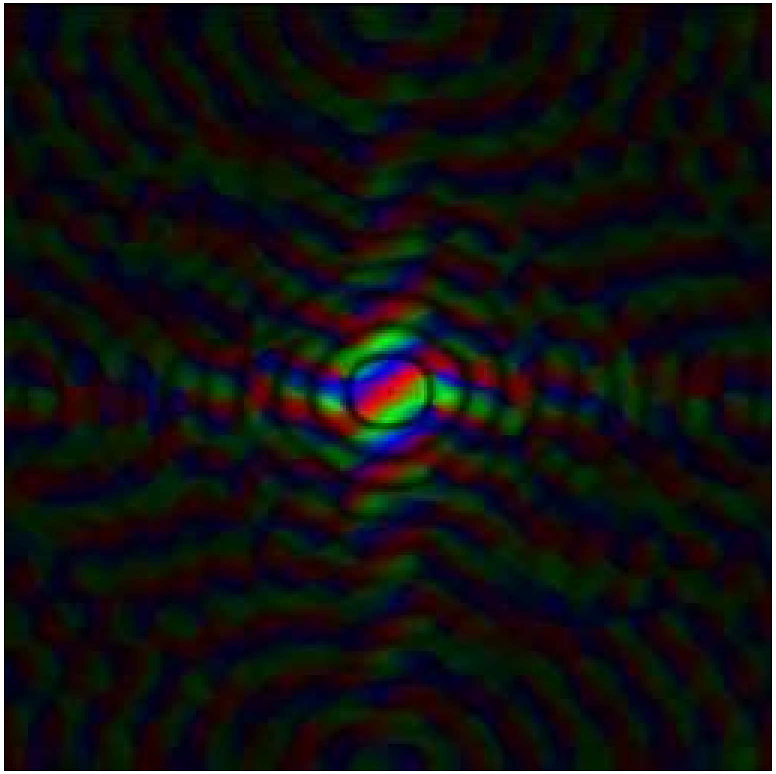


Full FFT
(Phase in Color)

FFT Image demo

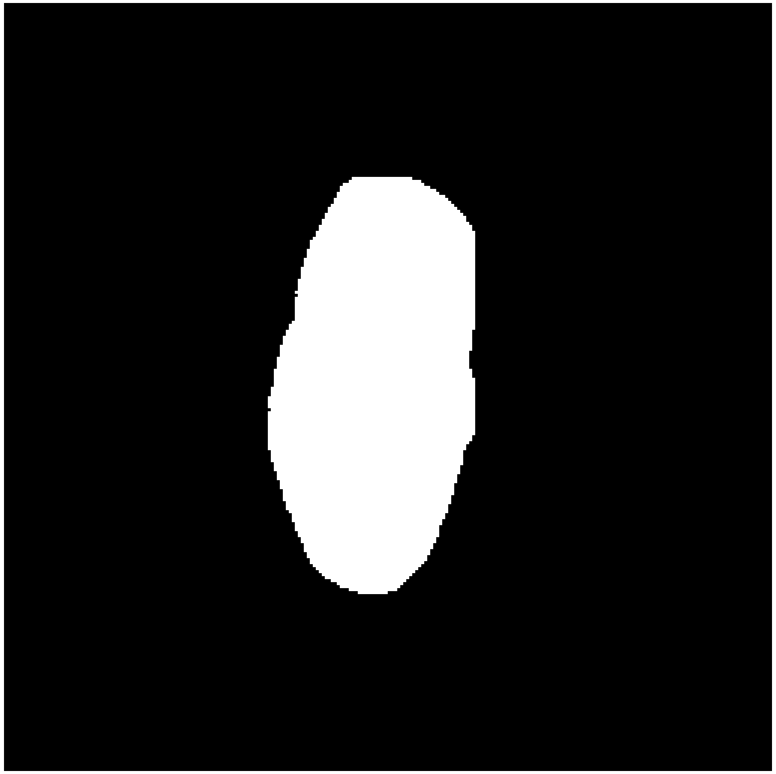


Real

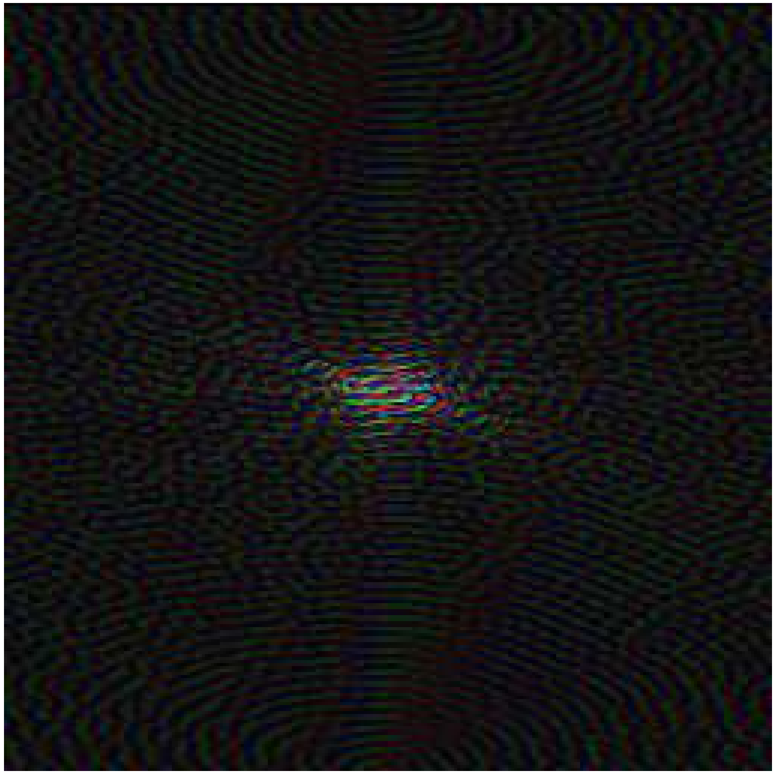


Full FFT
(Phase in Color)

FFT Image demo

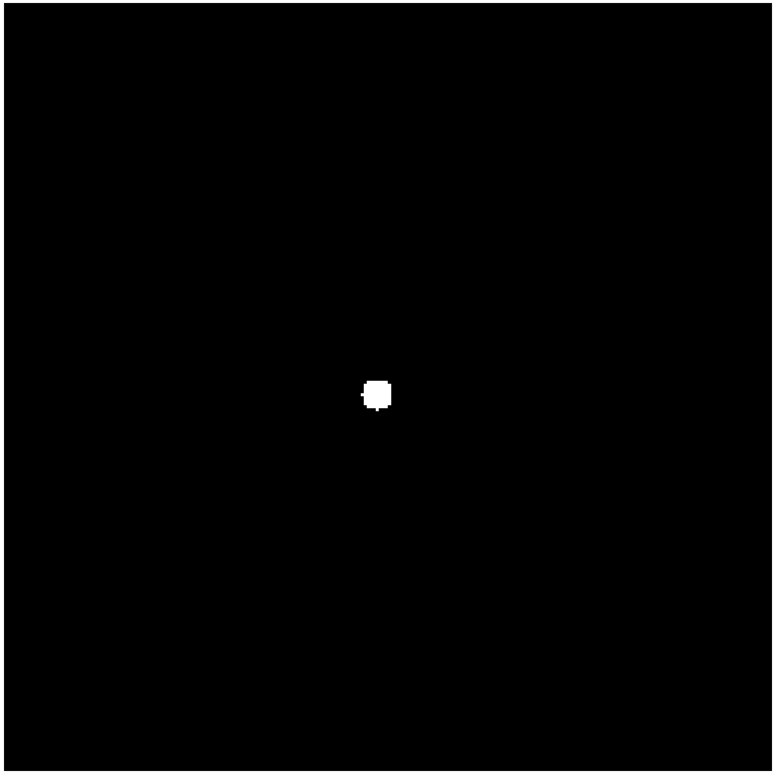


Real

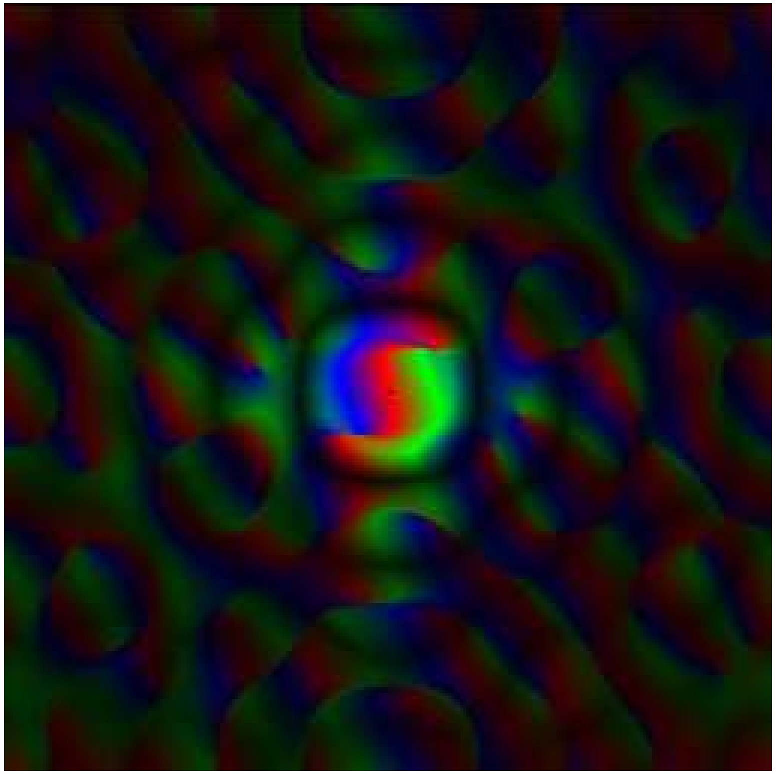


Full FFT
(Phase in Color)

FFT Image demo



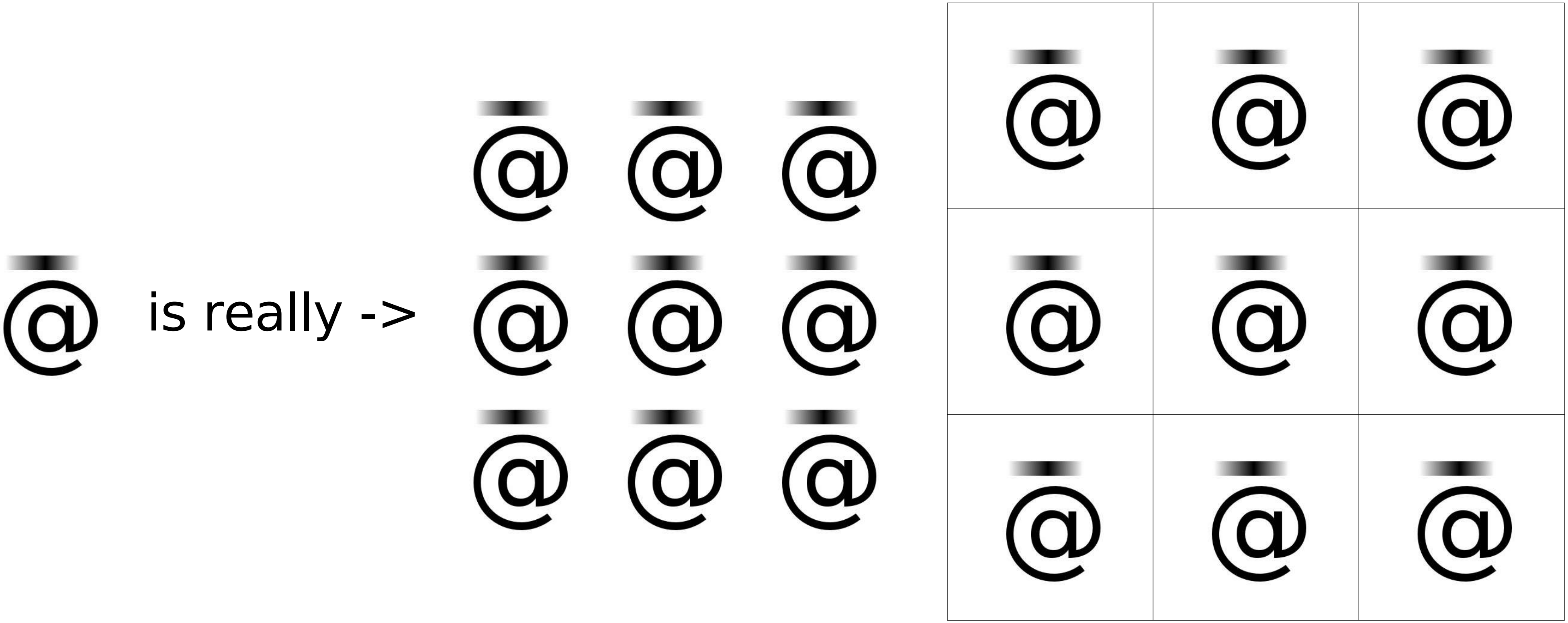
Real



Full FFT
(Phase in Color)

Infinite/Continuous vs. Finite/Discrete Fourier Transform

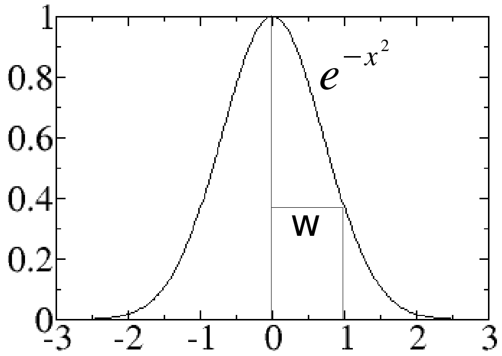
- Finite -> Periodic

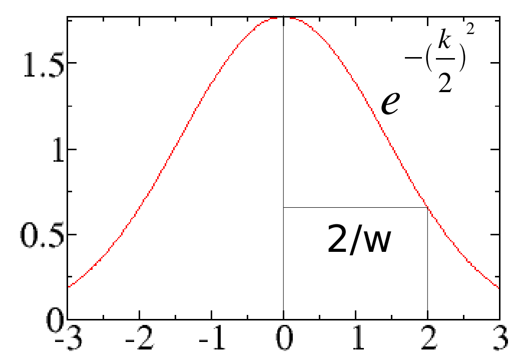


Gaussians

Gaussian or 'Normal' Distribution:

$$f(x) = e^{-\left(\frac{x}{w}\right)^2} \qquad \int_{-\infty}^{\infty} e^{ikx} e^{-\left(\frac{x}{w}\right)^2} dx = w\sqrt{\pi}\, e^{-\left(\frac{kw}{2}\right)^2}$$





Gaussians are everywhere

Fourier Convolution/Filtration

$$\bar{F}(k)=\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$h(x)=f(x)*g(x)=\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

$$\bar{H}(k)=\bar{F}(k)\bar{G}(k)$$

Test Image

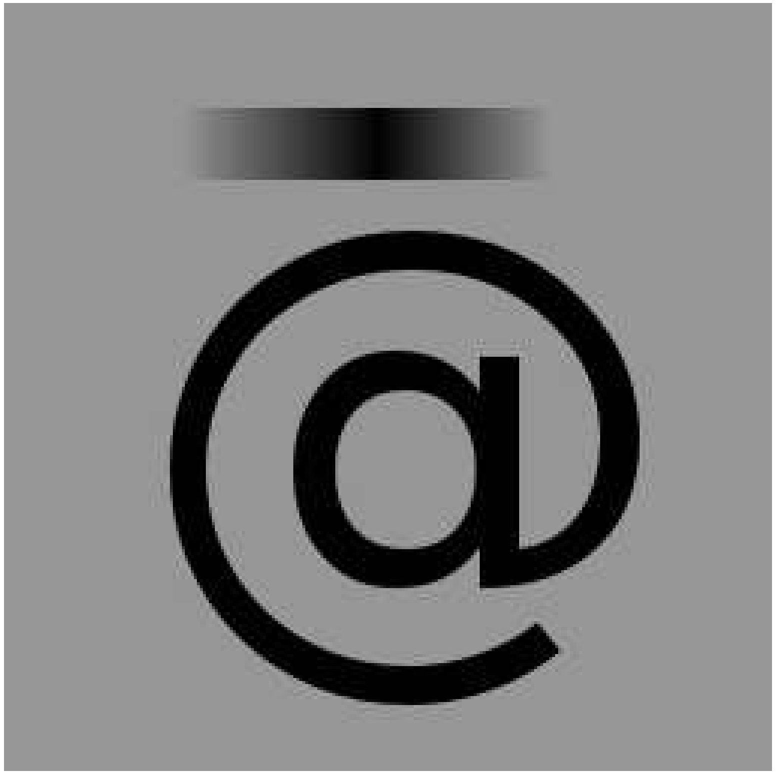
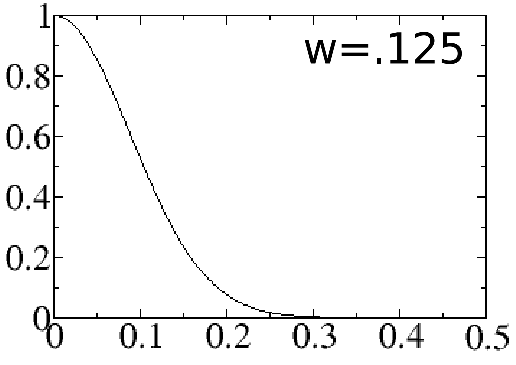


Image Filtration
Gaussian Lowpass



$$e^{-\left(\frac{x}{w}\right)^2}$$

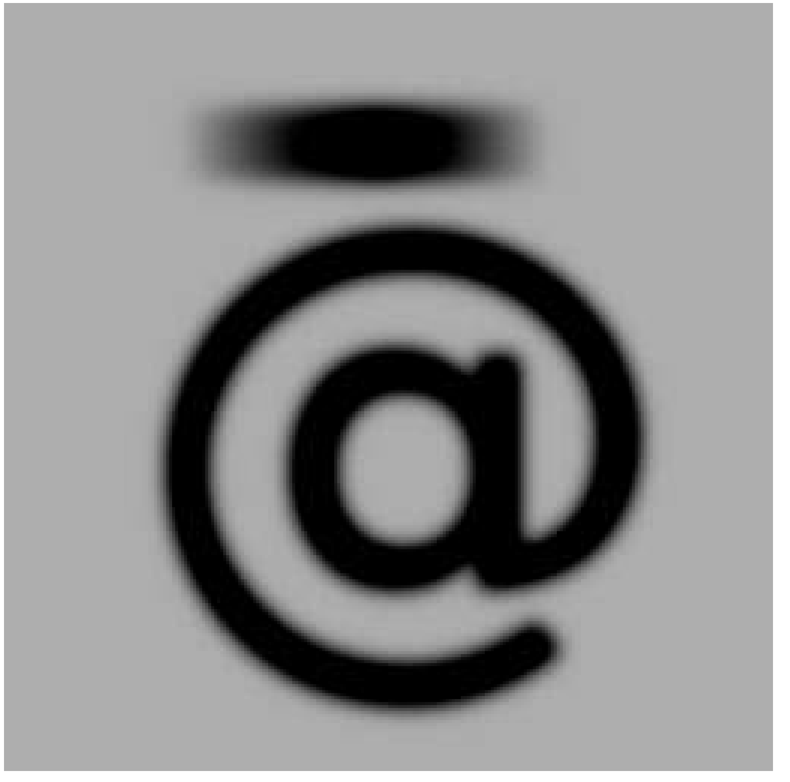
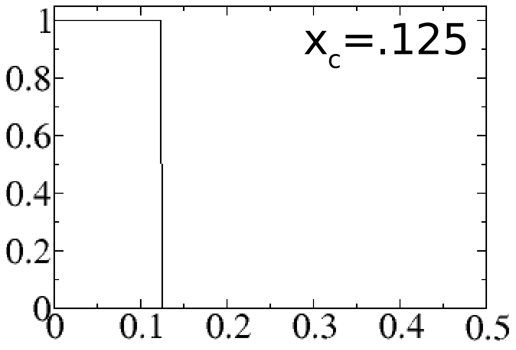
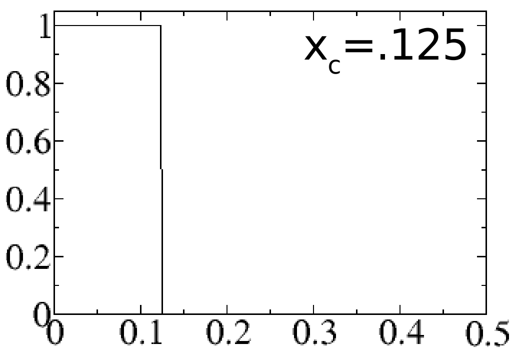


Image Filtration
Sharp Lowpass



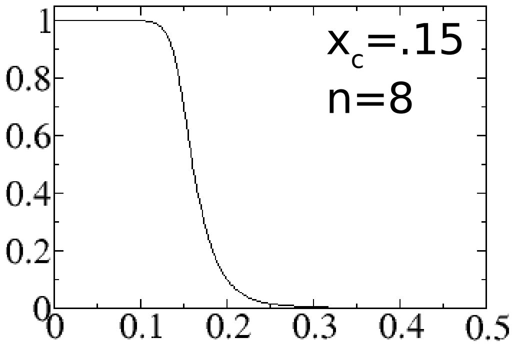
$x<x_c$ -> 1.0
else -> 0

Image Filtration
Sharp Lowpass



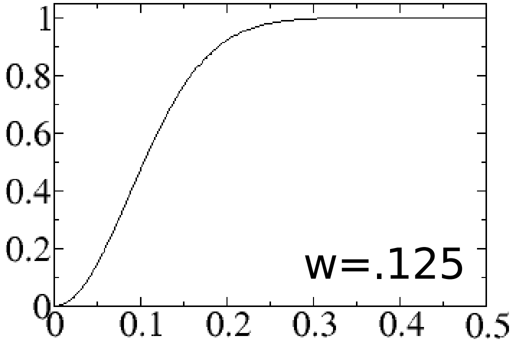
$x<x_c$ -> 1.0
else -> 0

Image Filtration
Butterworth Lowpass



$$\frac{1}{\sqrt{1+\left(\frac{x}{x_c}\right)^{2n}}}$$

Image Filtration
Gaussian Highpass

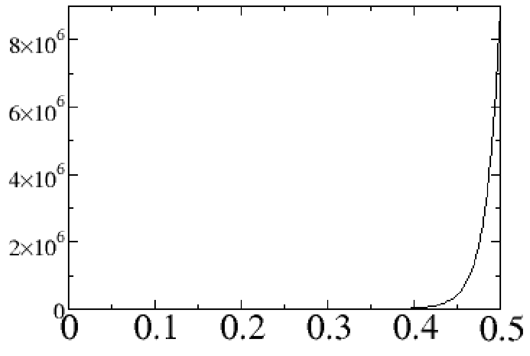


$$1.0 - e^{-\left(\frac{x}{w}\right)^2}$$

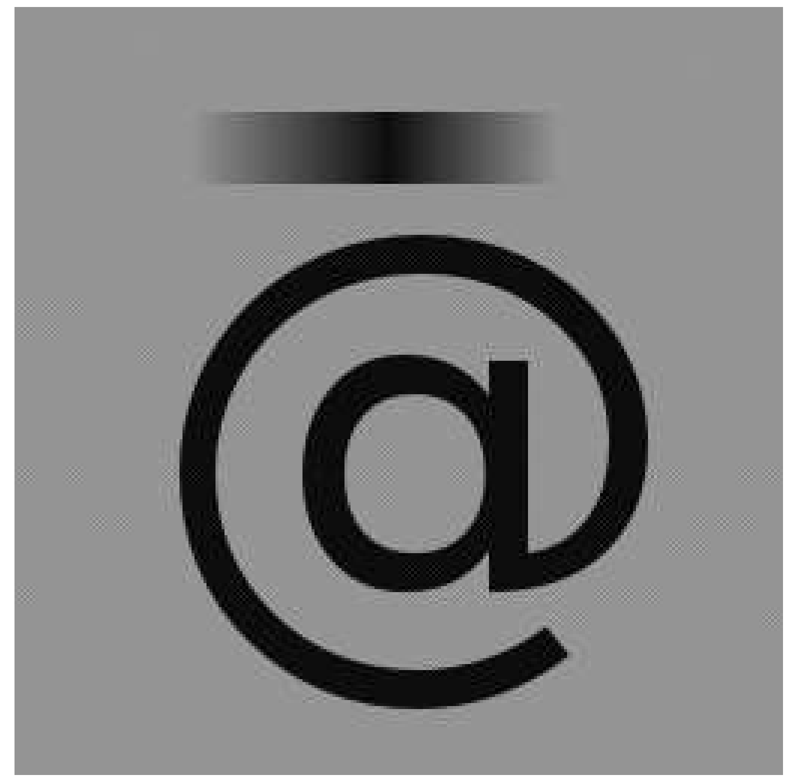
Deconvolution

$$e^{-(\frac{x}{w})^2}$$

Deconvolution

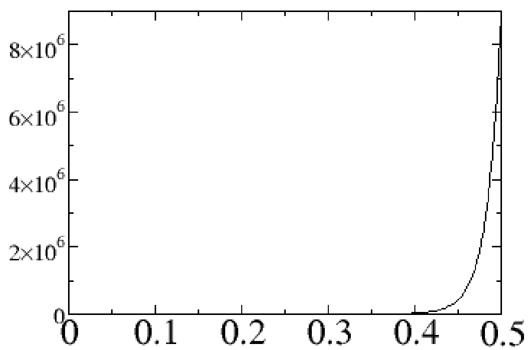


$$e^{\left(\frac{x}{w}\right)^2}$$



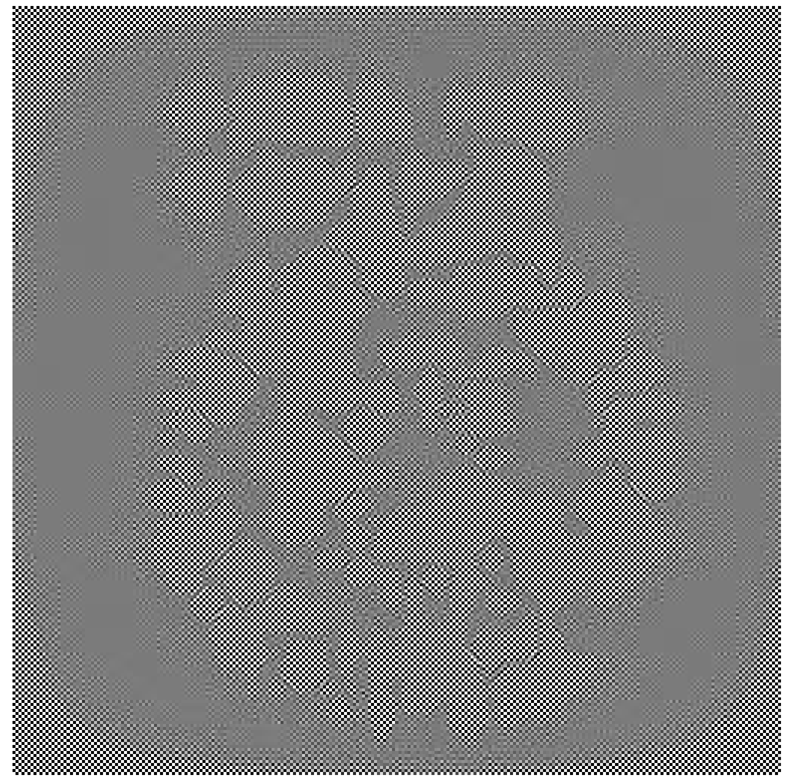
@

Deconvolution

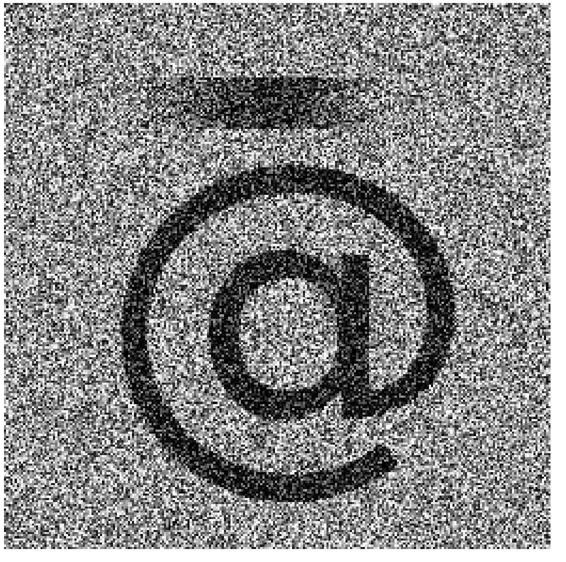


$$e^{\left(\frac{x}{w}\right)^2}$$

From Discrete valued image

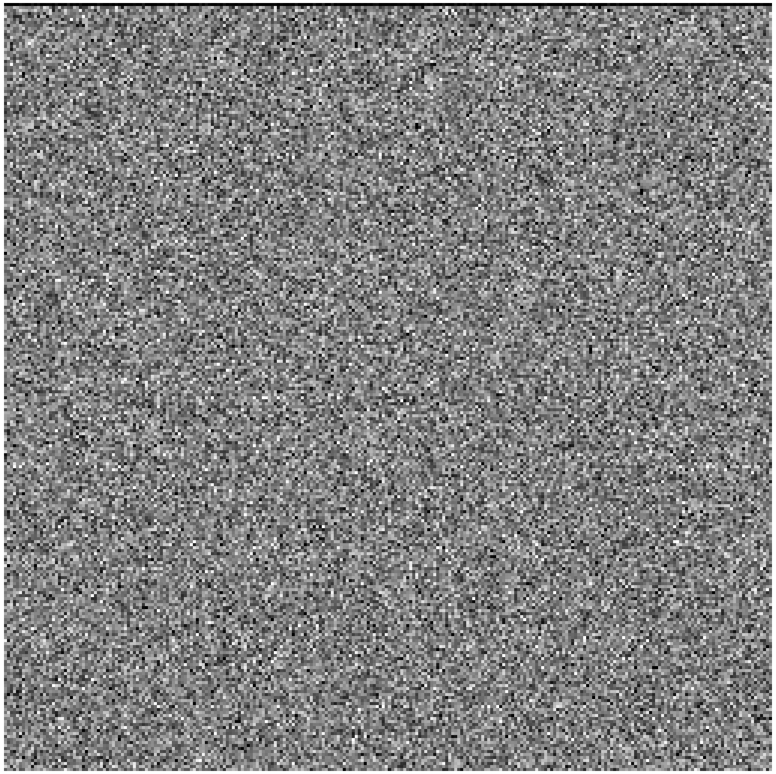


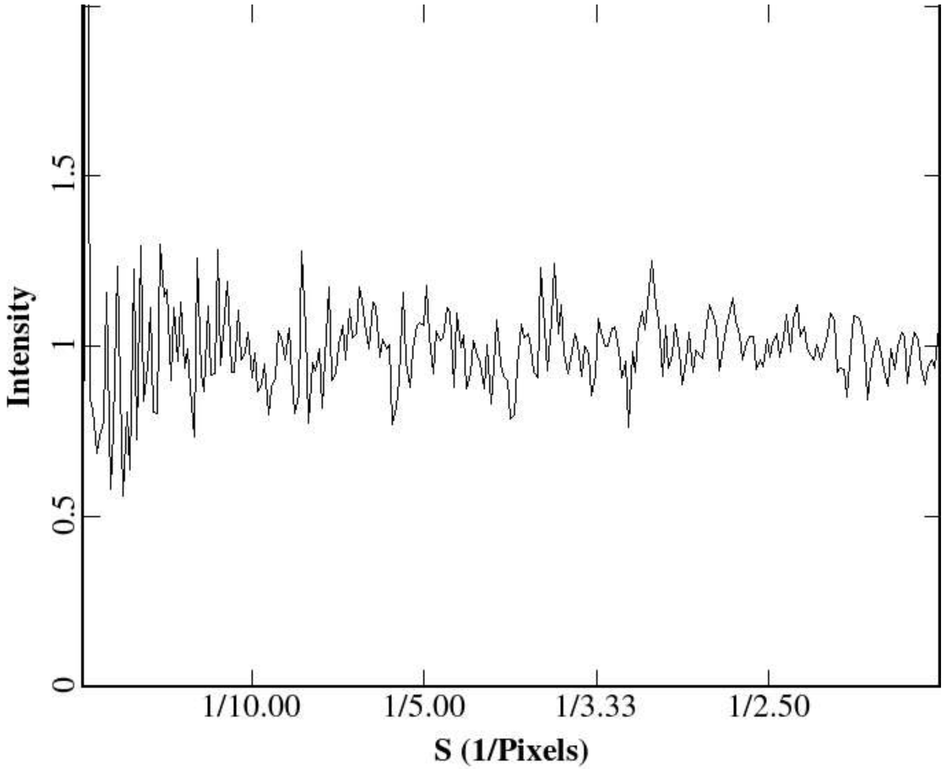
Noise



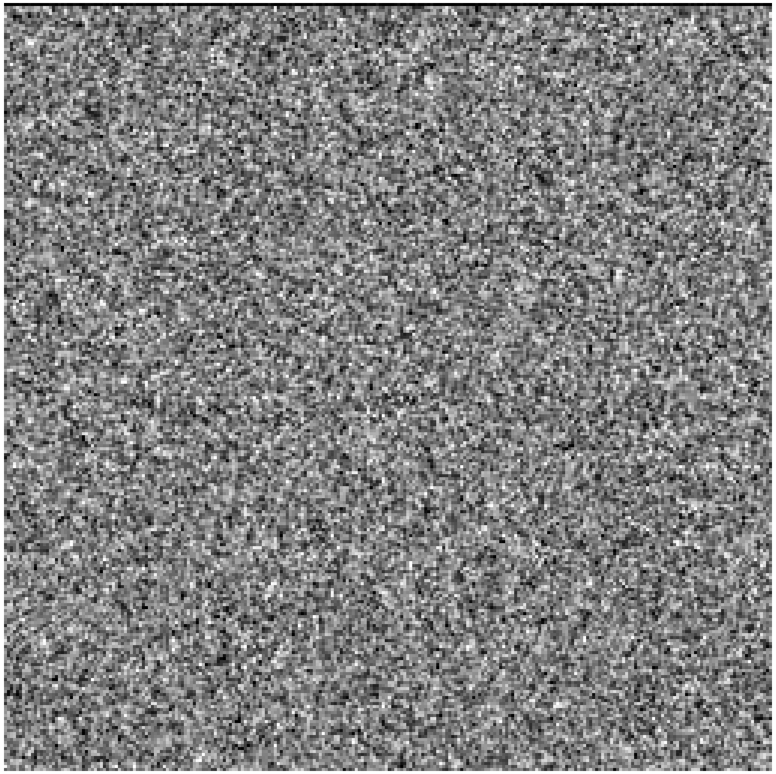
How do we characterize the noise ?

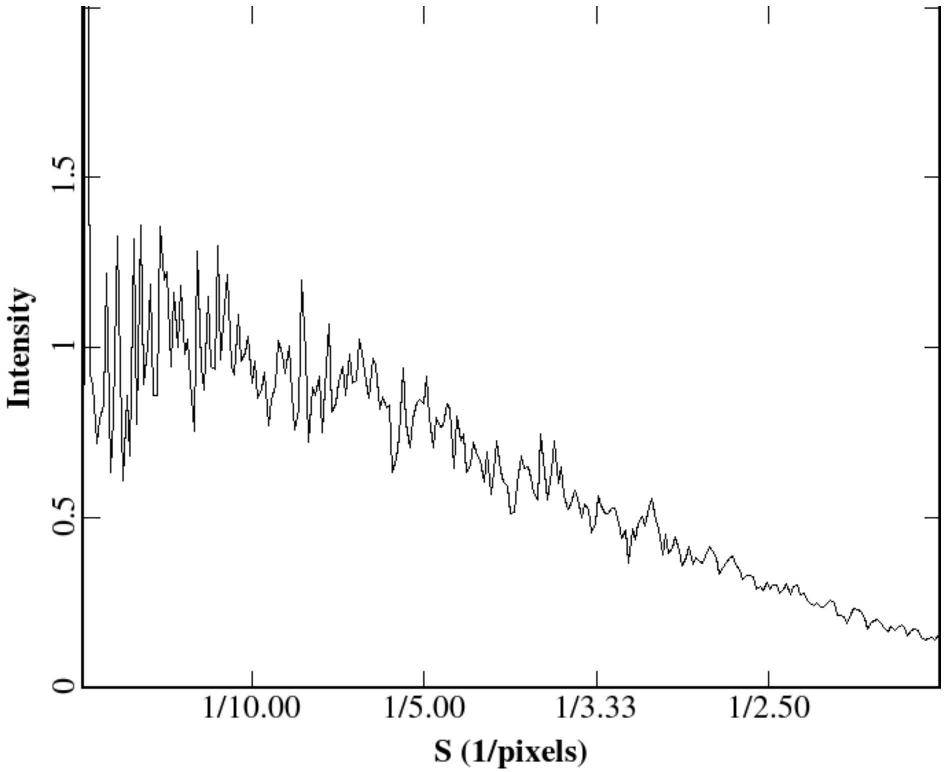
White Noise



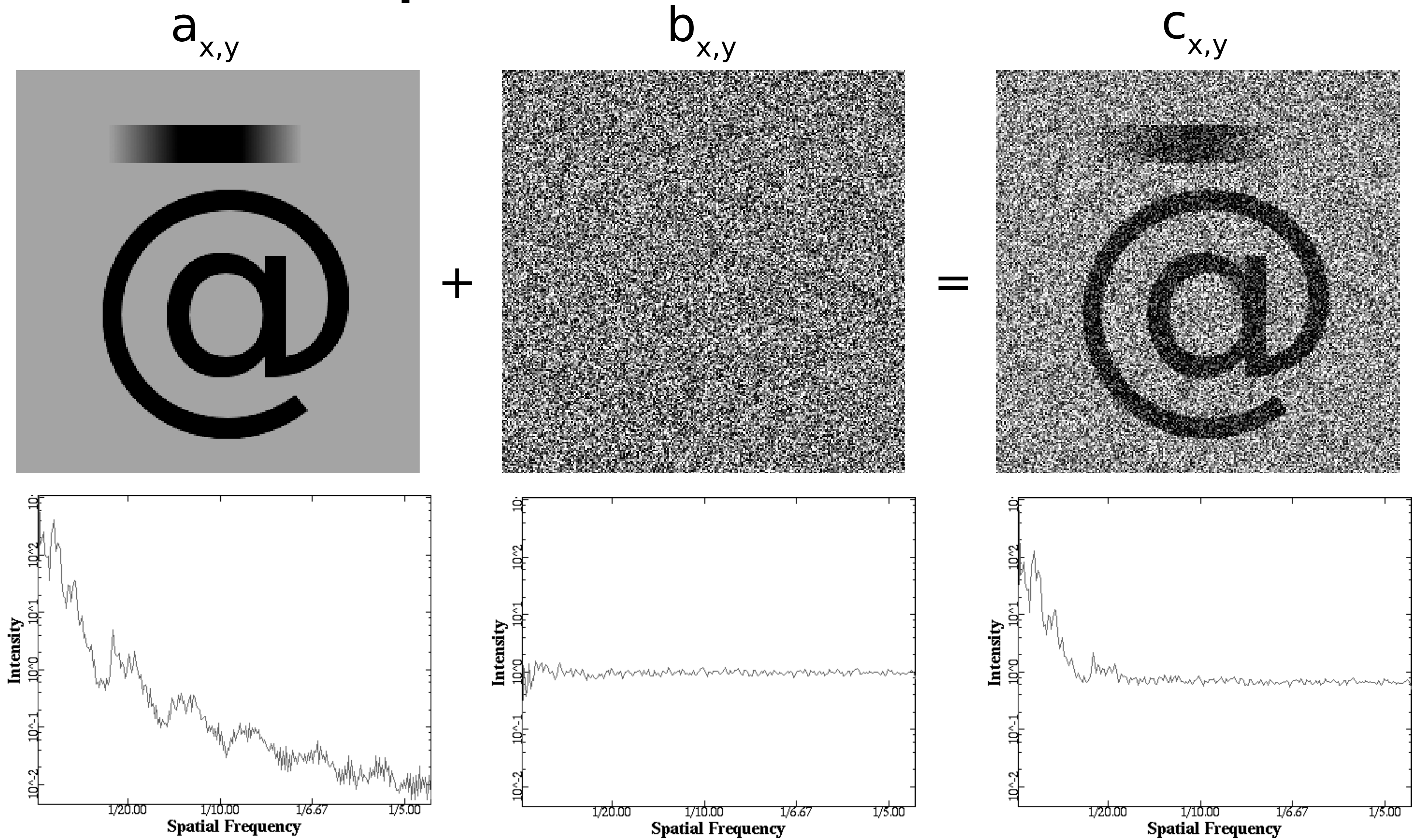


Pink Noise





Optimal Filtration



Optimal Filtration

$$W(s_x, s_y) = F(s_x, s_y)\, C(s_x, s_y)$$

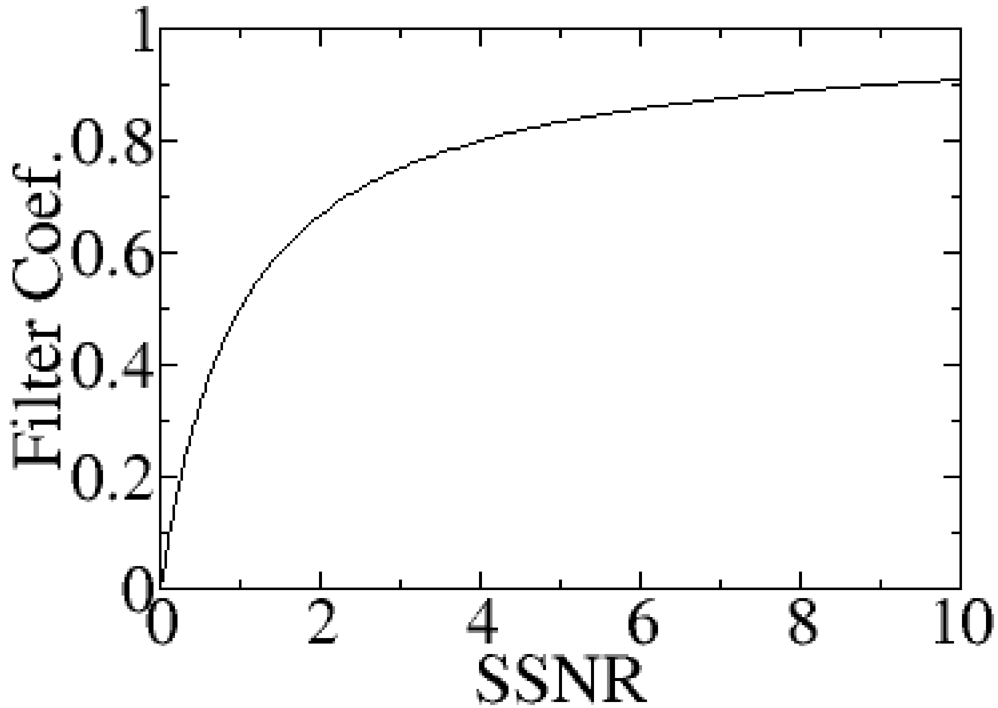
where

$$\sum_{x,y} (c_{x,y} - a_{x,y})^2$$

Answer is a Wiener Filter:

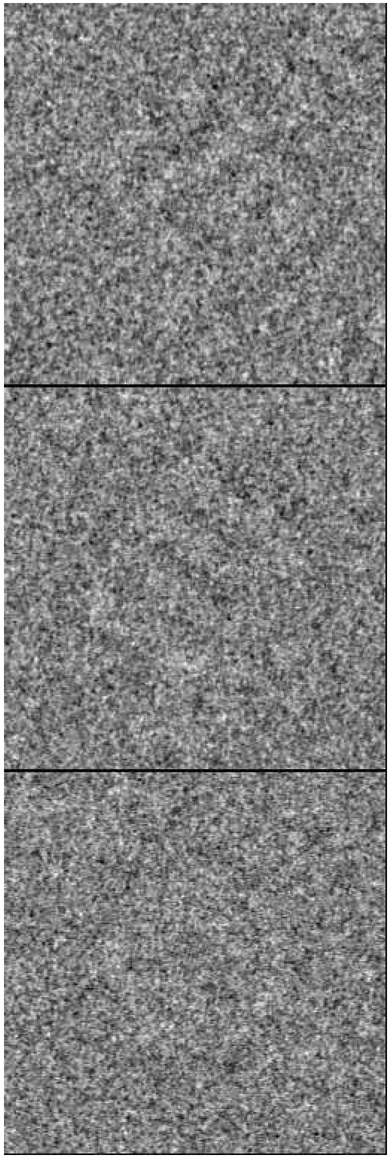
$$F(s_x, s_y) = \frac{1}{1 + \frac{1}{\mathrm{SNR}(s_x, s_y)}}$$

Wiener Filter

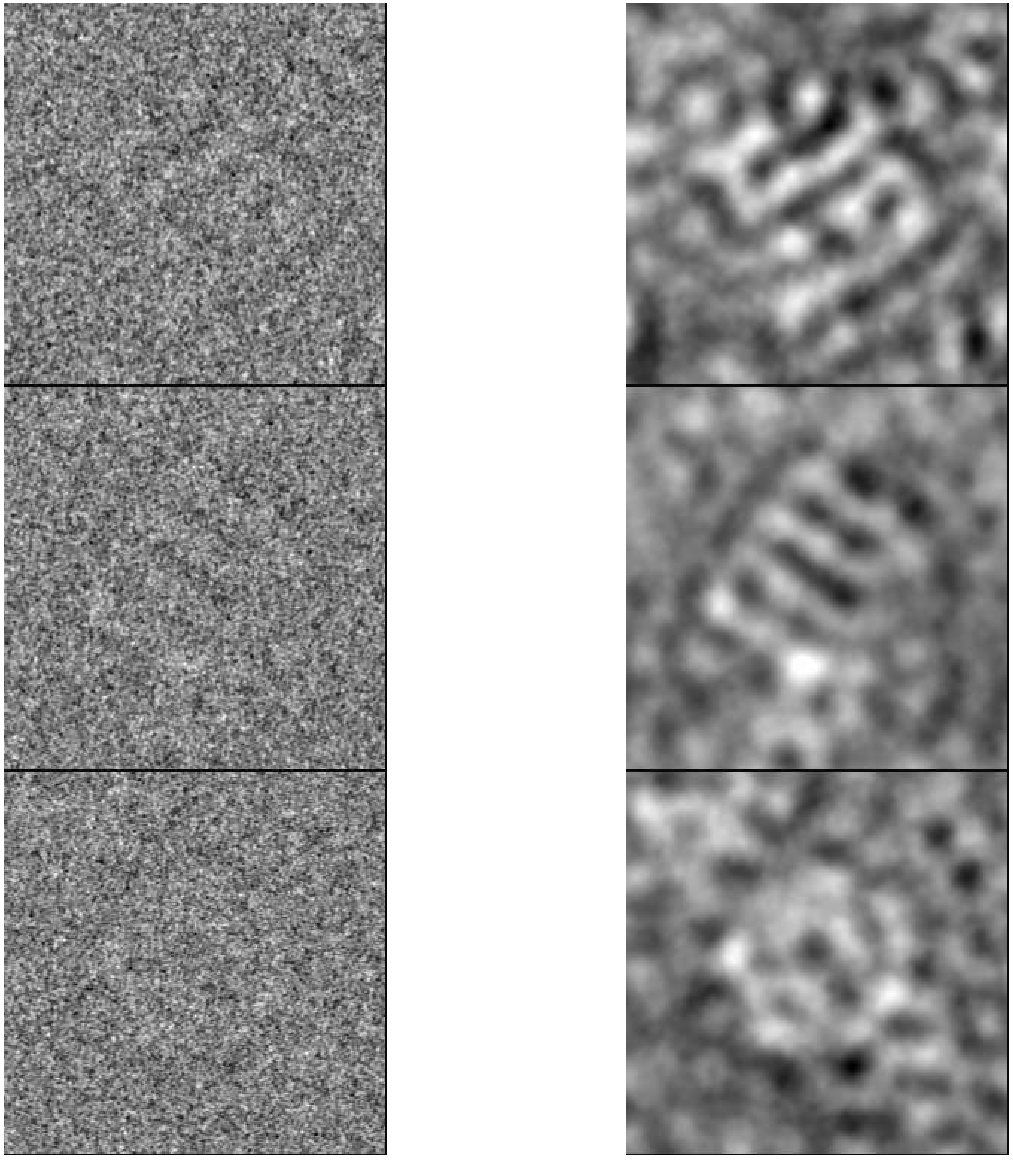


$$F(s_x, s_y) = \frac{1}{1 + \frac{1}{\mathrm{SNR}(s_x, s_y)}}$$

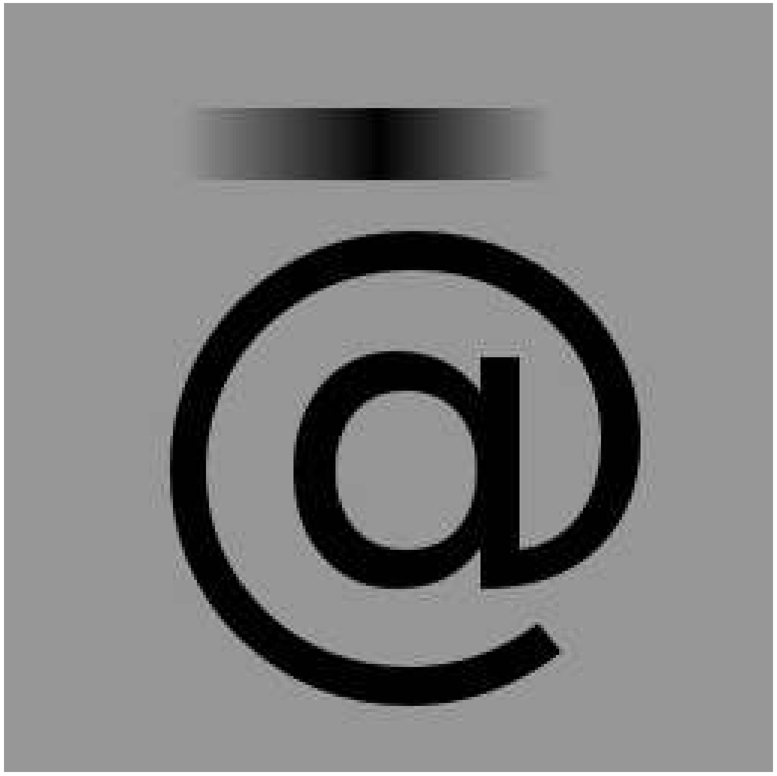
Wiener Filter



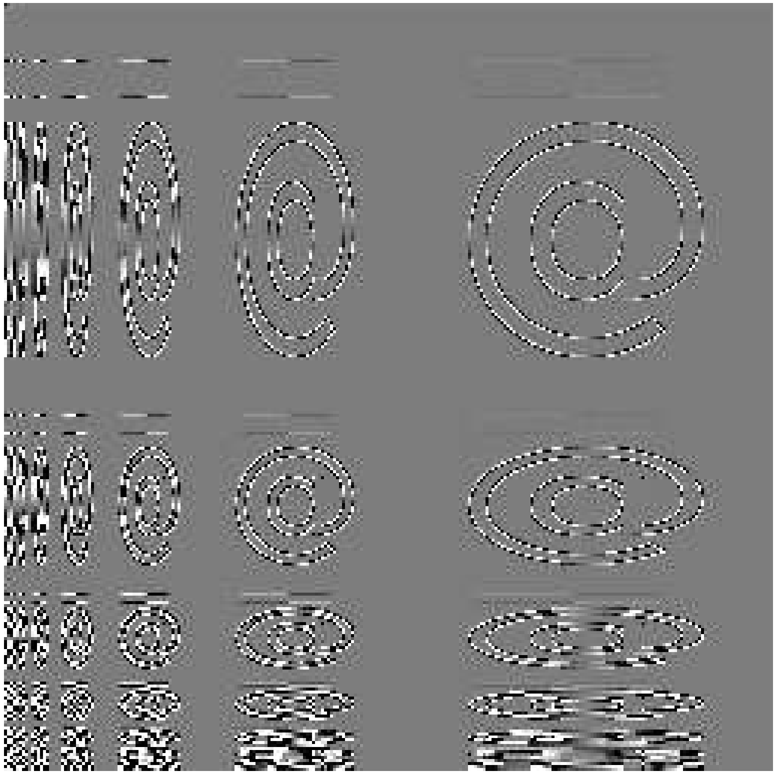
Wiener Filter



Wavelets in 2-D

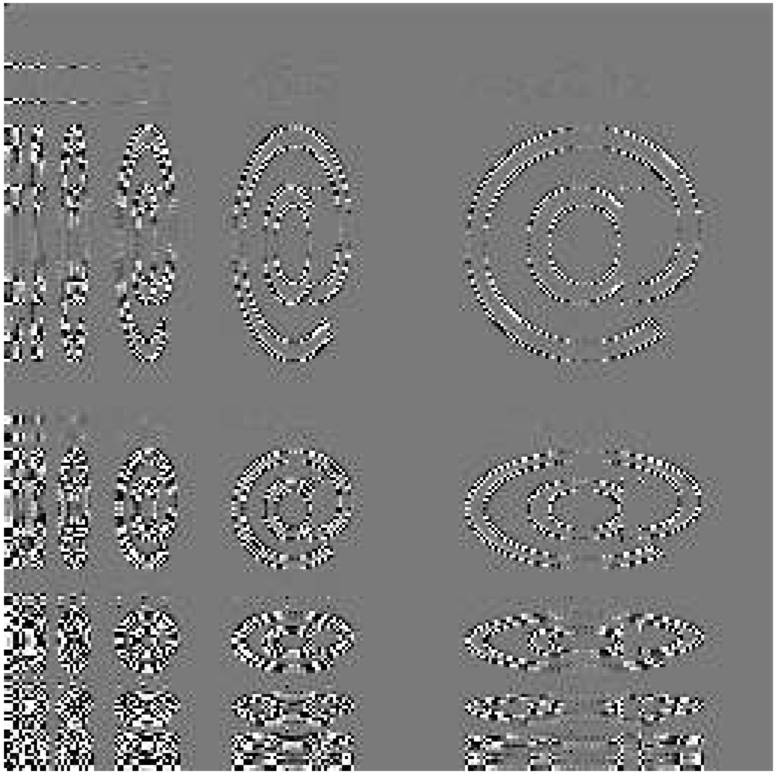


Wavelets in 2-D



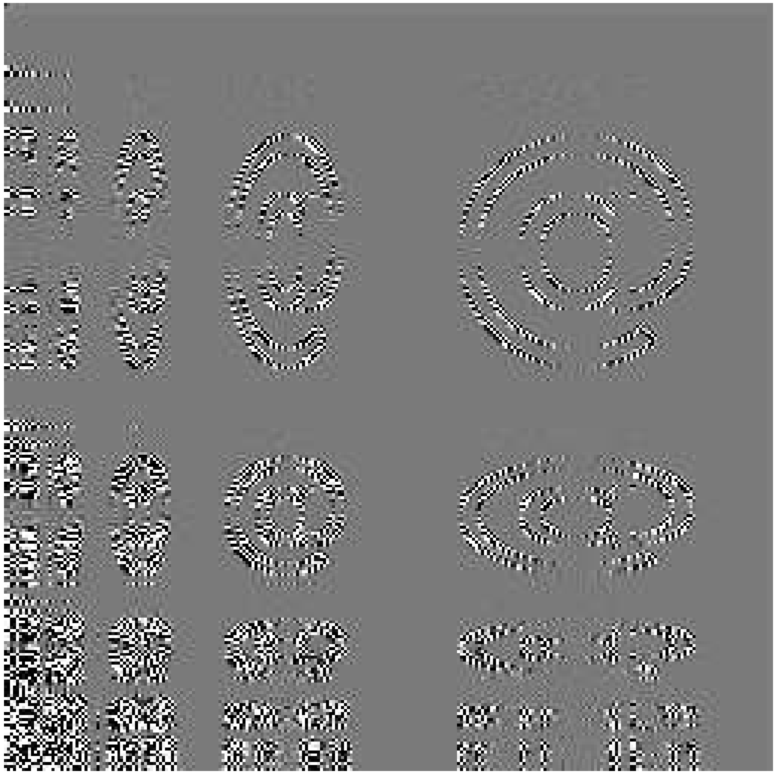
Harr

Wavelets in 2-D



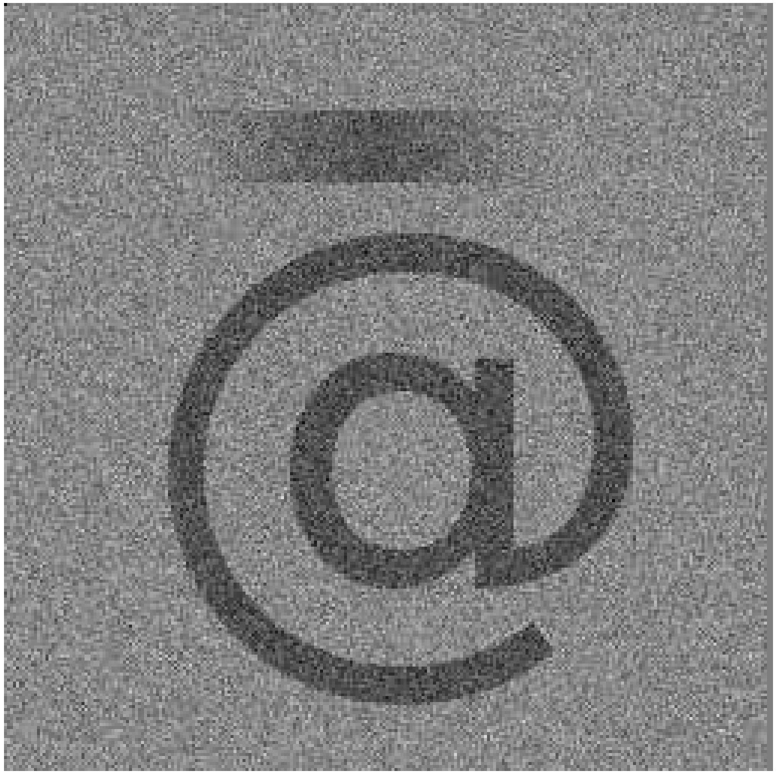
Daub4

Wavelets in 2-D

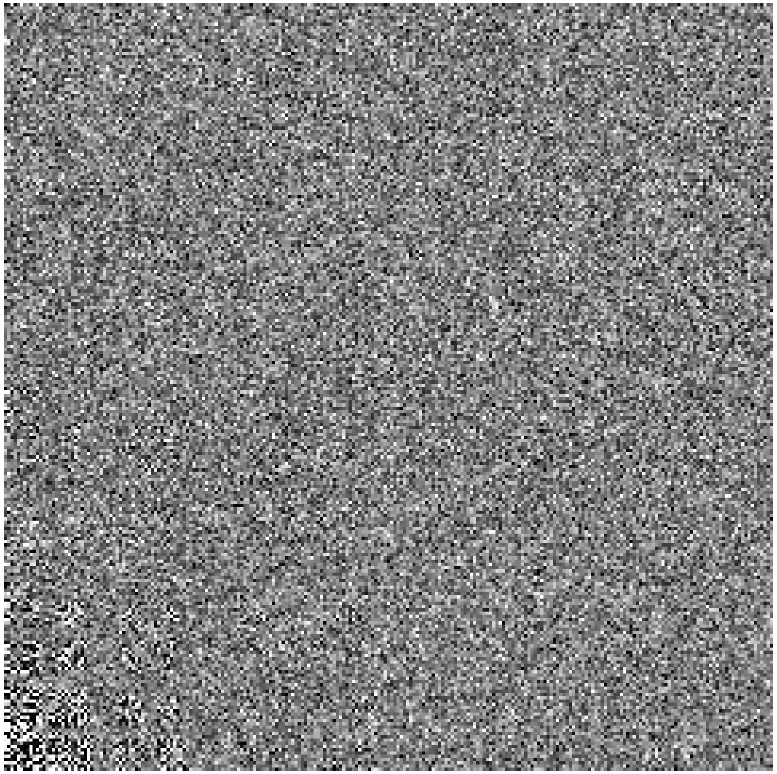


Daub8

Wavelets in 2-D

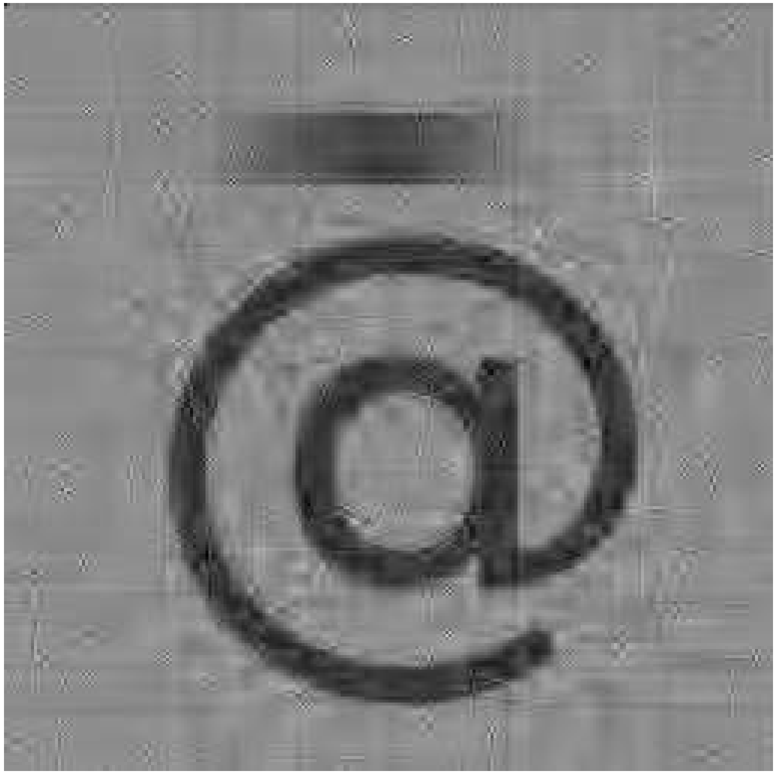


Wavelets in 2-D



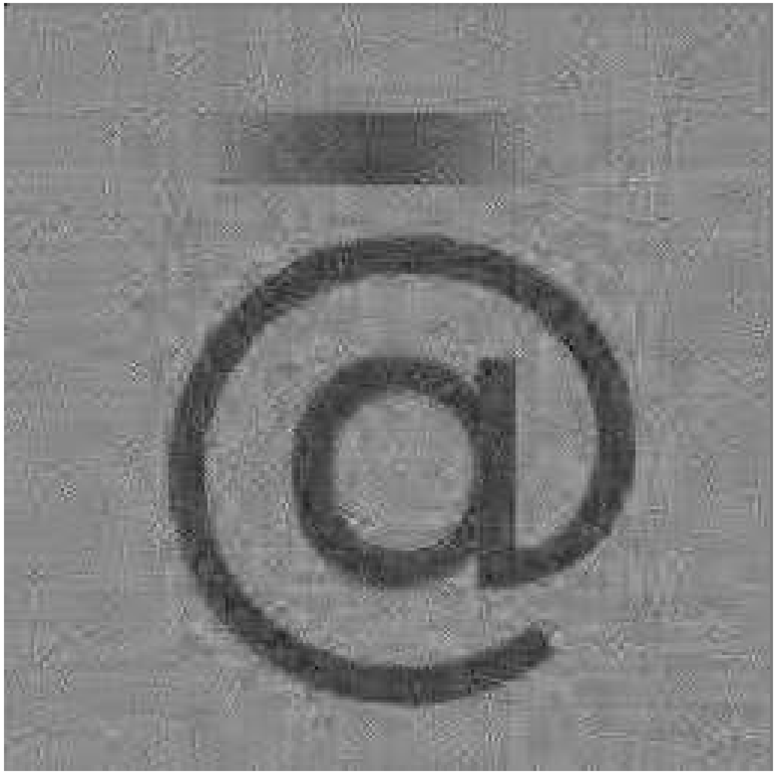
Daub8

Wavelets in 2-D



Strong threshold
-thr<x<thr -> 0

Wavelets in 2-D



Less strong threshold (thr smaller)
-thr<x<thr -> 0

Transformation

- Translation
- Rotation
- Scaling
- Skewing
- Non-linear transformations (tensors)

Orthogonal
Similarity
Affine

Affine Transformations

$$x' = x\, m_{00} + y\, m_{01} + m_{02}$$

$$y' = x\, m_{10} + y\, m_{11} + m_{12}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Combine transformations by matrix multiplication

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

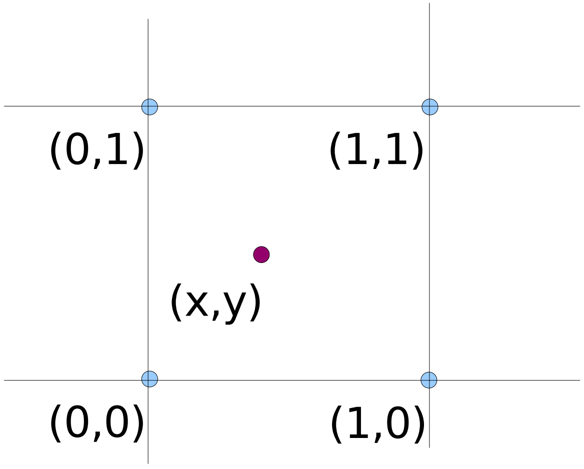
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformation -> Interpolation

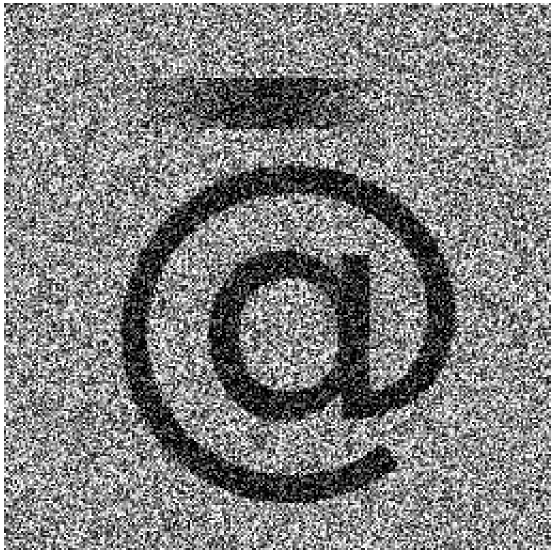
- Bilinear
- Bicubic Splines
- Gaussian
- Fourier
- Arbitrary Kernel ?

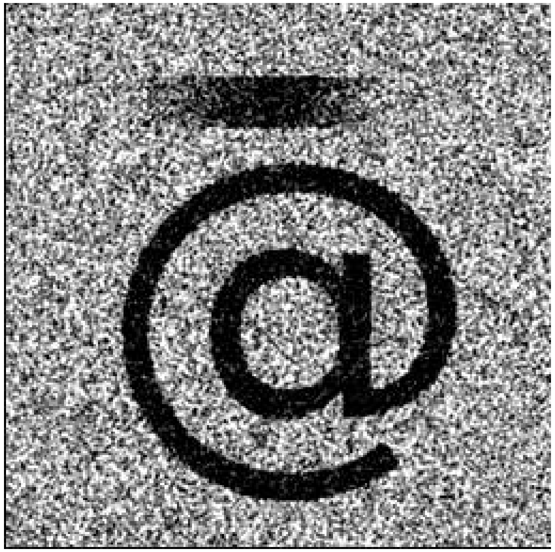


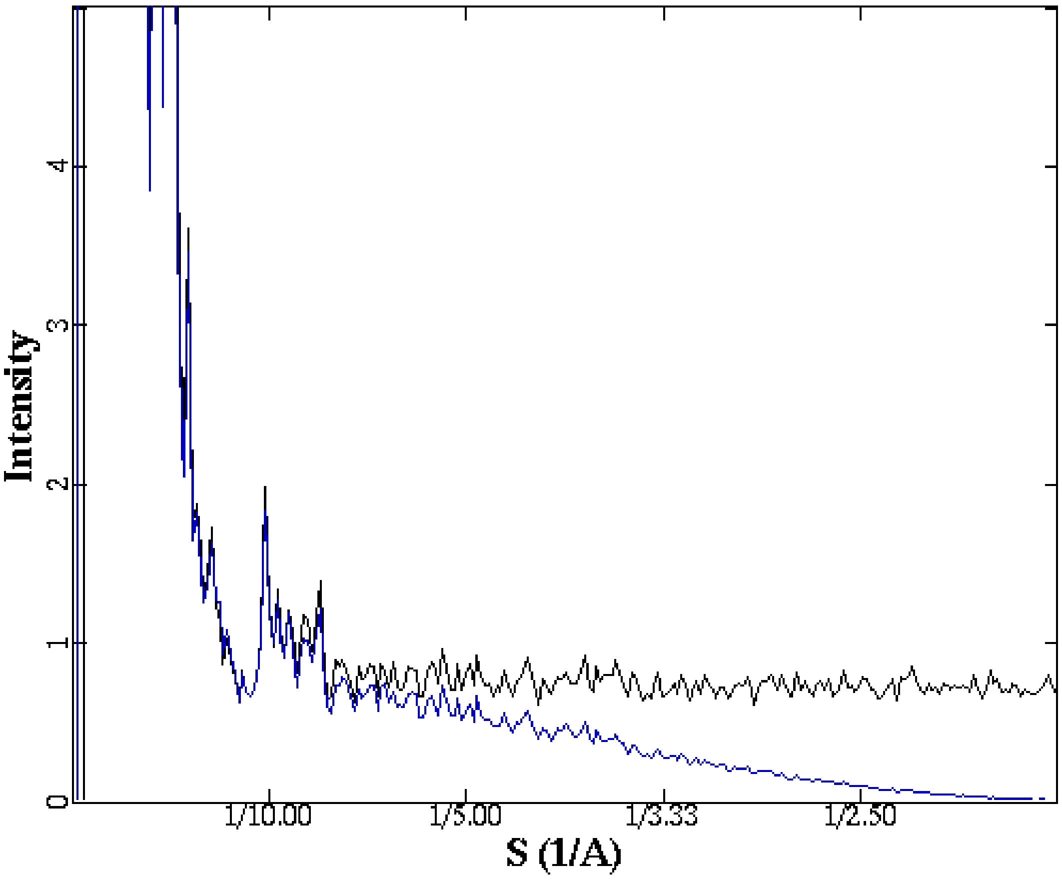
Bilinear:

$$v_{xy} = x\,y\,v_{00} + x(1-y)v_{10} + (1-x)\,y\,v_{01} + (1-x)(1-y)v_{11}$$

Interpolation Effects





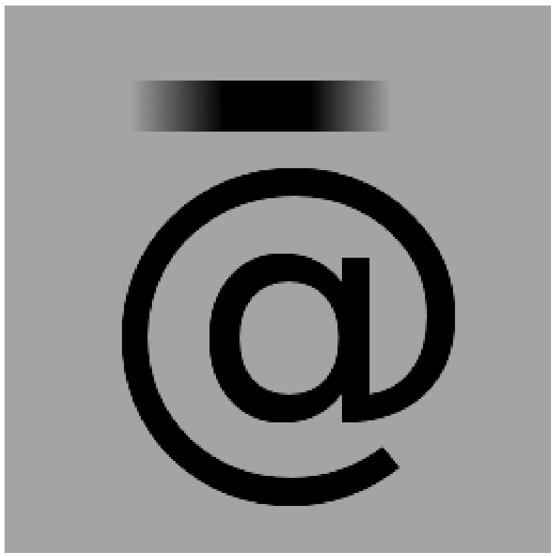


Measures of Similarity

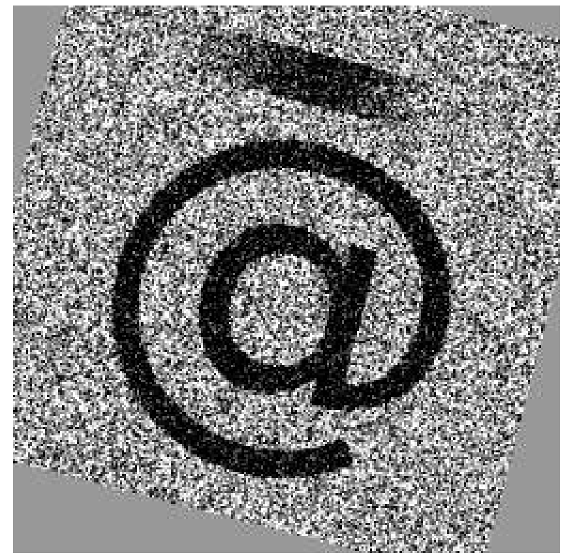
- Correlation Coefficient
- Variance (transformed density)
- Variance (matched filter)
- Phase Residual
- Mutual Information
- etc.

Correlation

$a_{x,y}$



$b_{x,y}$



$$\frac{1}{\sqrt{\sum_{x,y} a_{x,y}^2 \cdot \sum_{x,y} b_{x,y}^2}} \sum_{x,y} a_{x,y} b_{x,y}$$

Normalized from -1.0 - 1.0

Registration

- Exhaustive search
- Minimization techniques
- Cross correlation

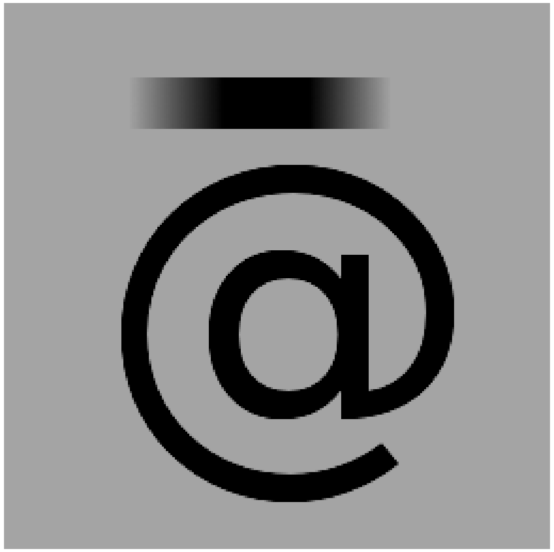
Cross Correlation

$$c(x) = \text{Corr}(f(x), g(x)) = \int_{-\infty}^{\infty} f(x+t)\, g(t)\, dt$$

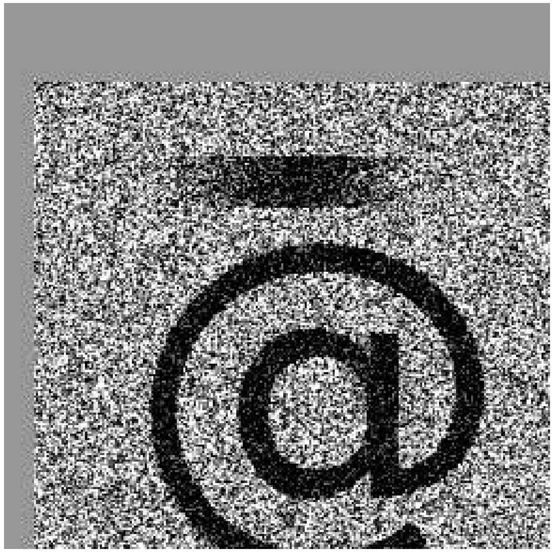
In Fourier Space:

$$C(k) = F(k)\, G^{*}(k)$$

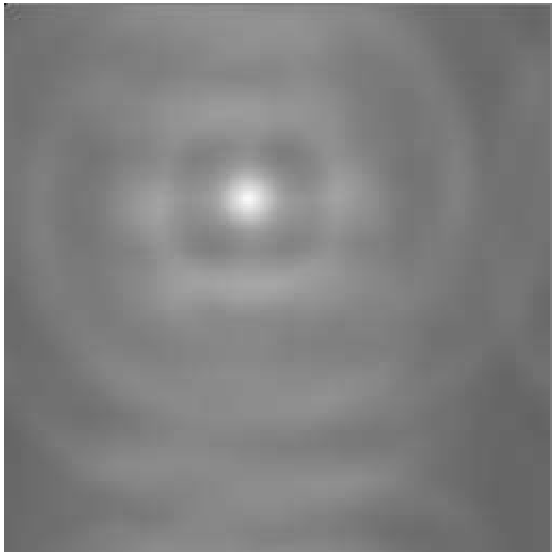
$f_{x,y}$



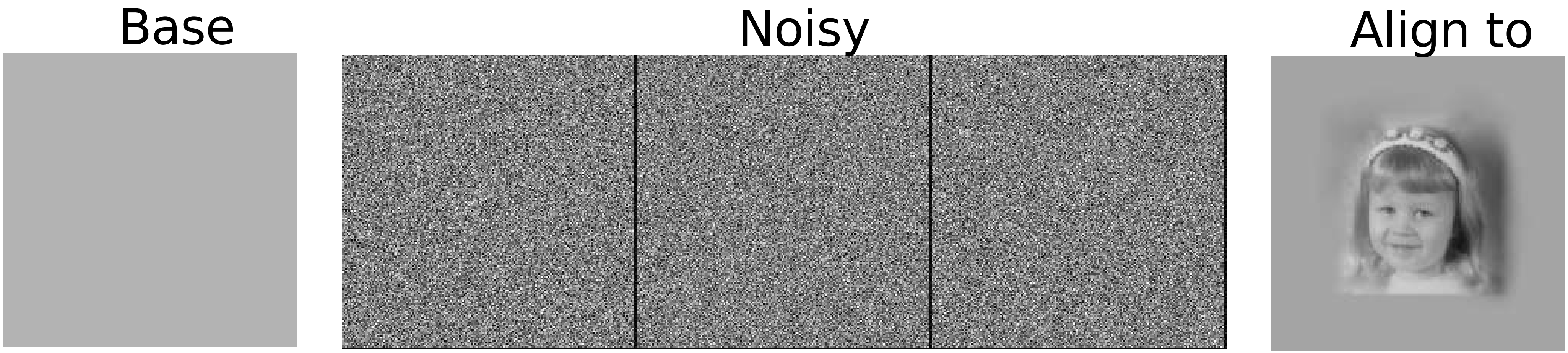
$g_{x,y}$



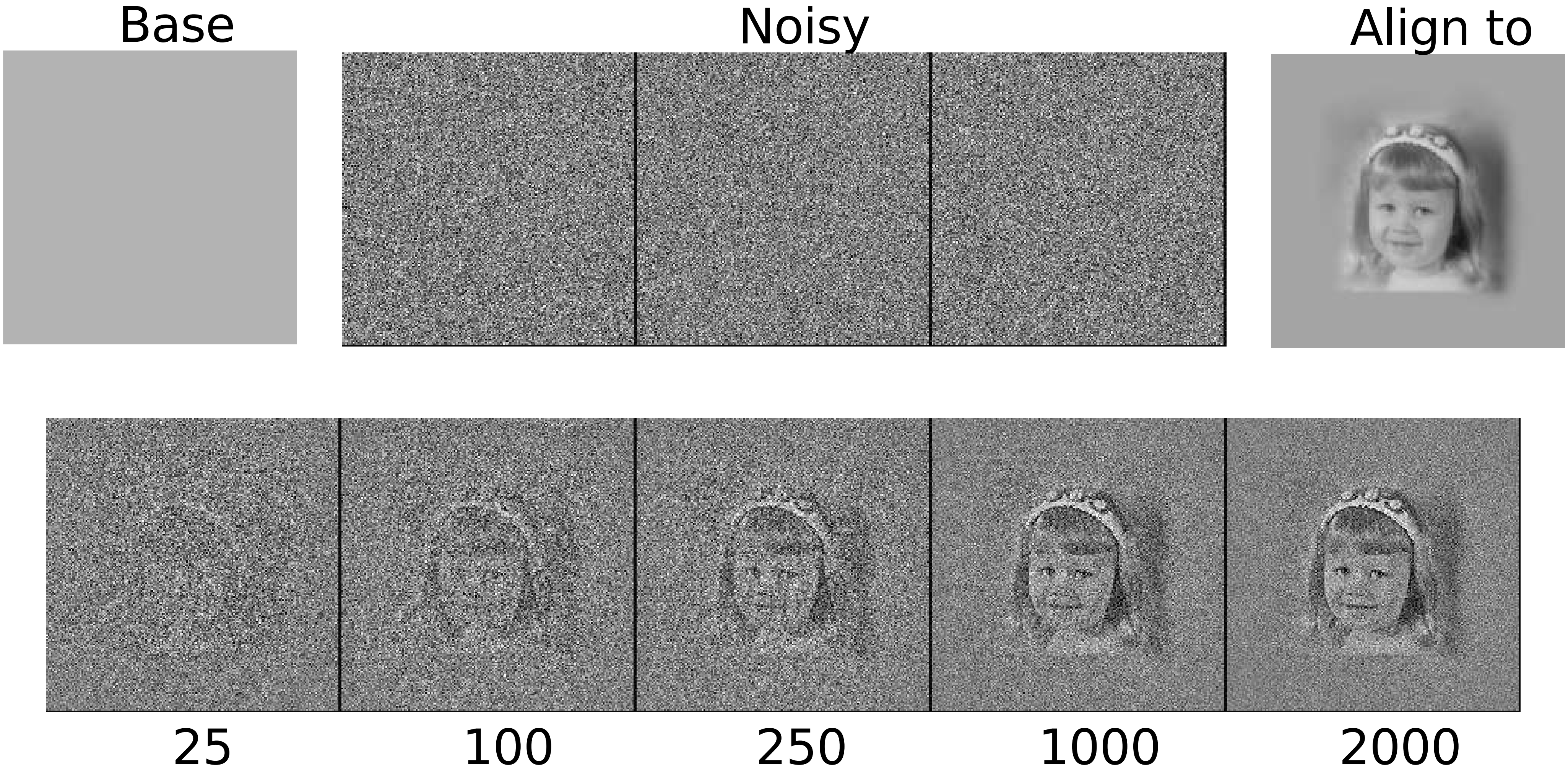
$c_{x,y}$



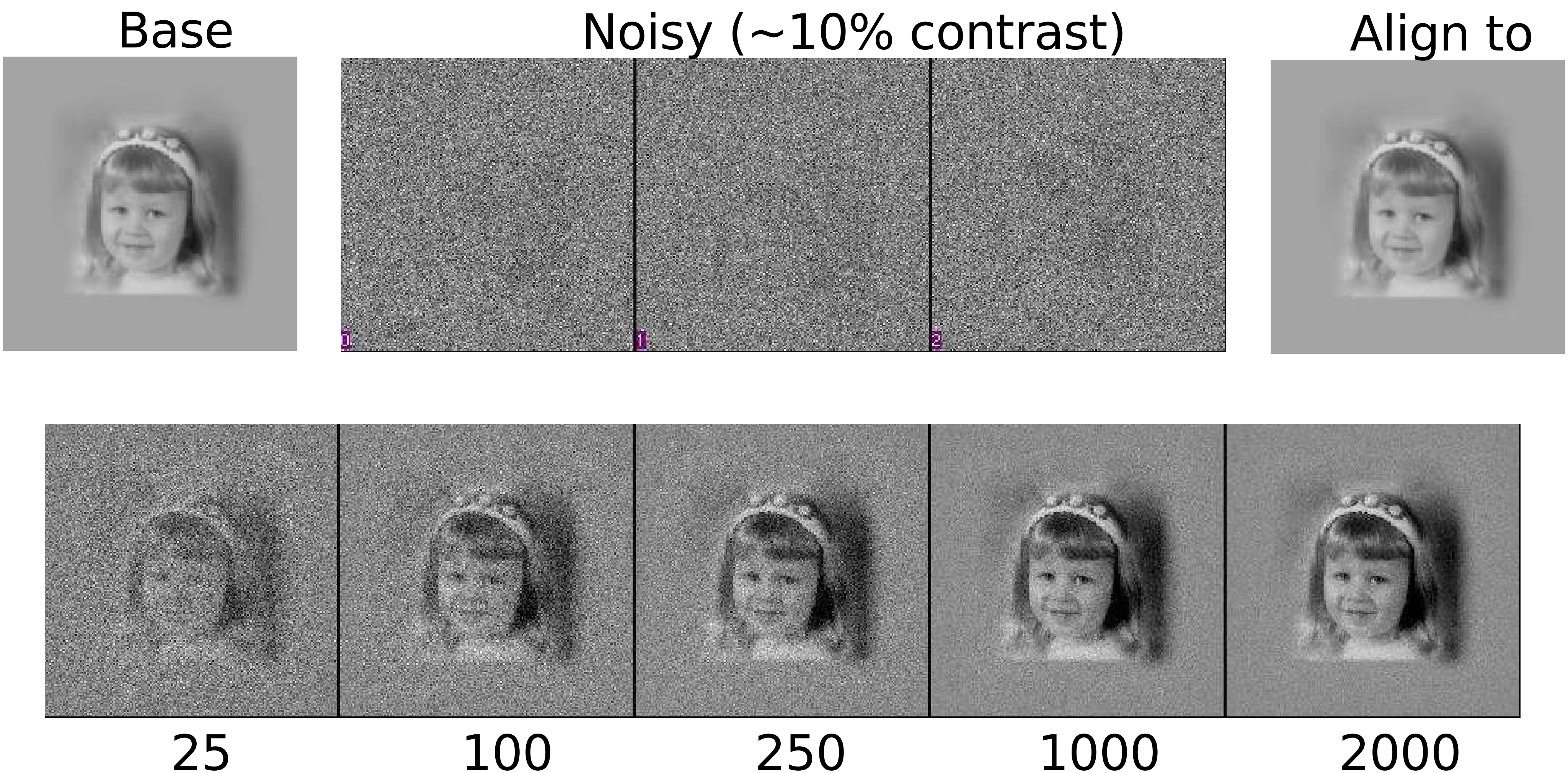
Model Bias



Model Bias



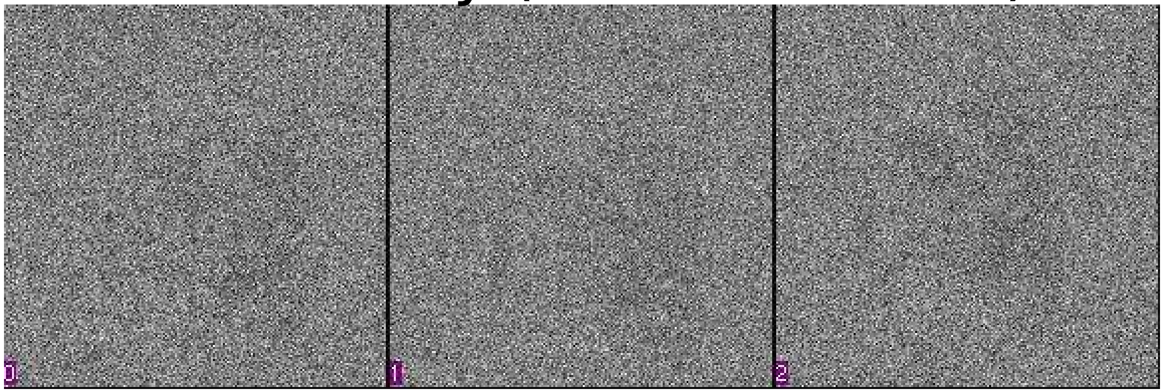
Model Bias



Model Bias

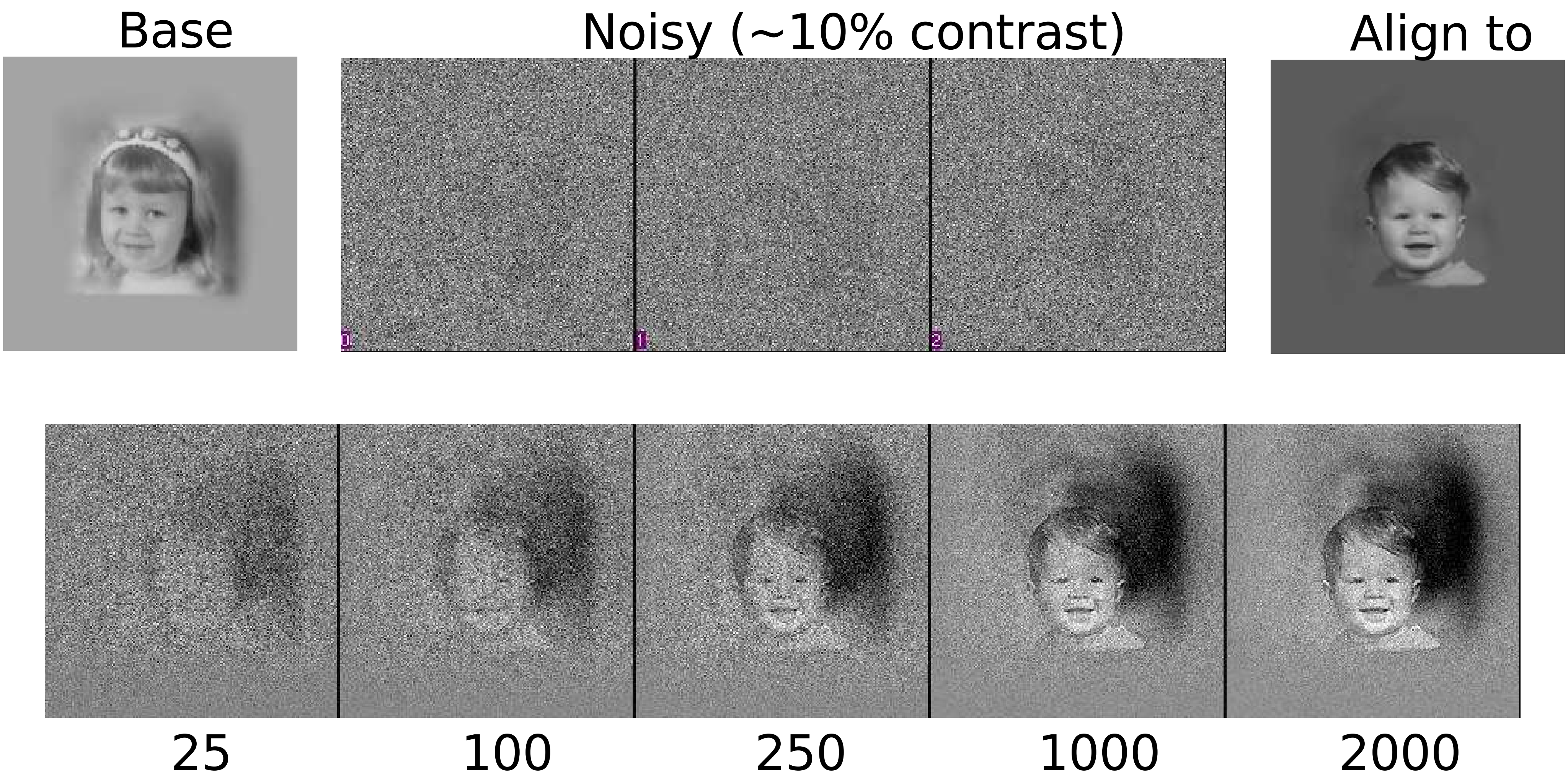
Base

Noisy (~10% contrast)

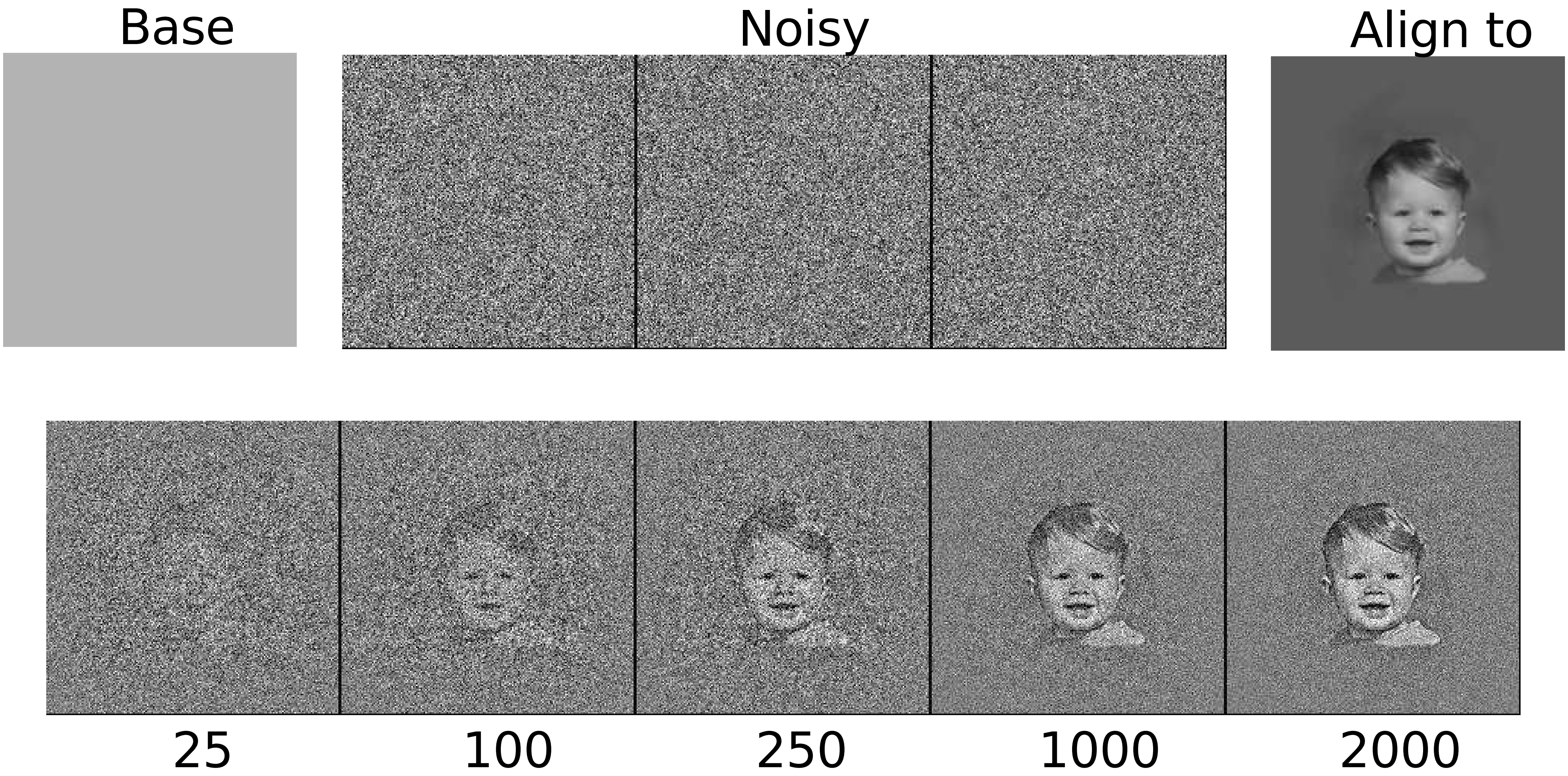


Align to

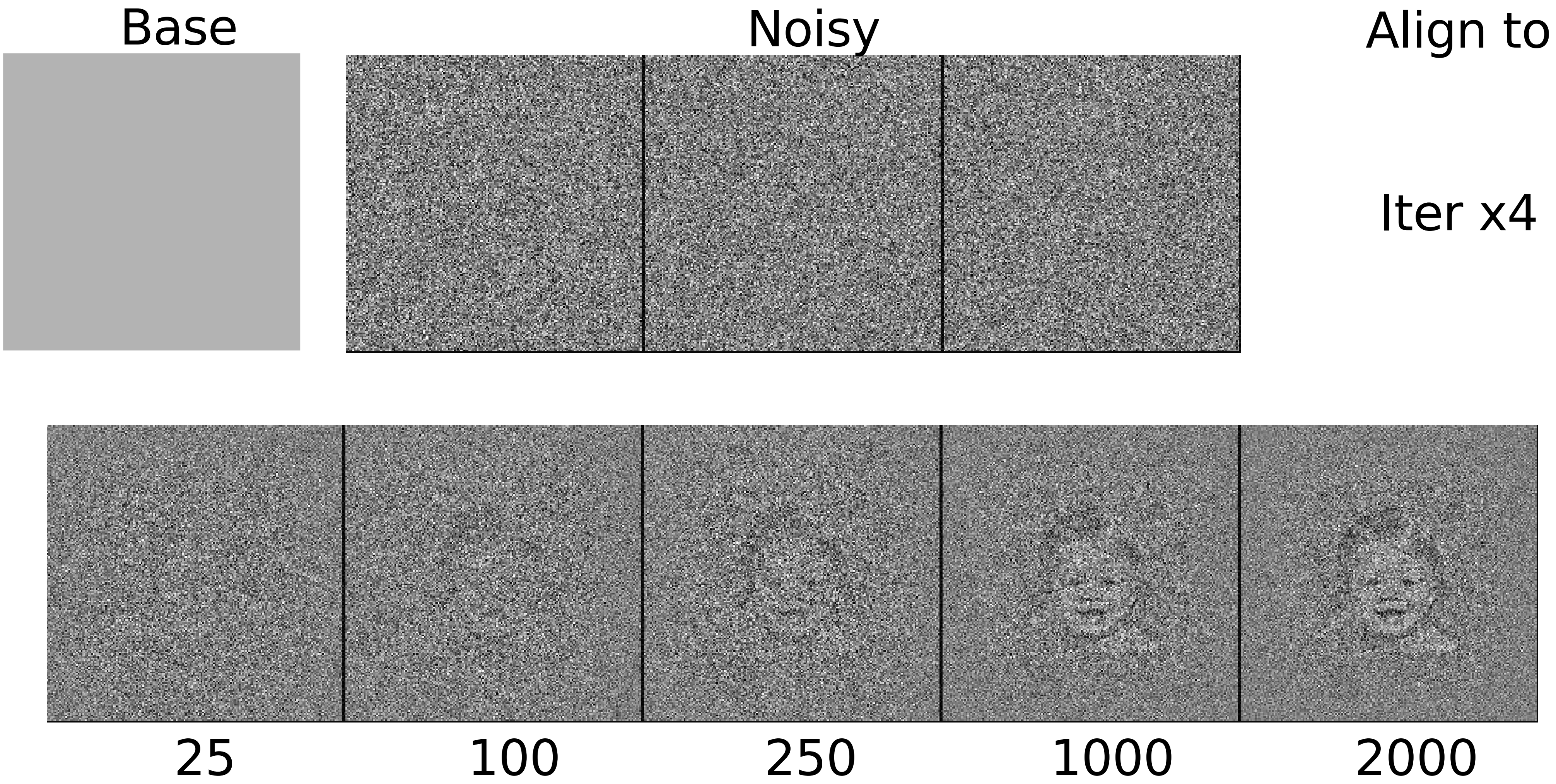
Model Bias



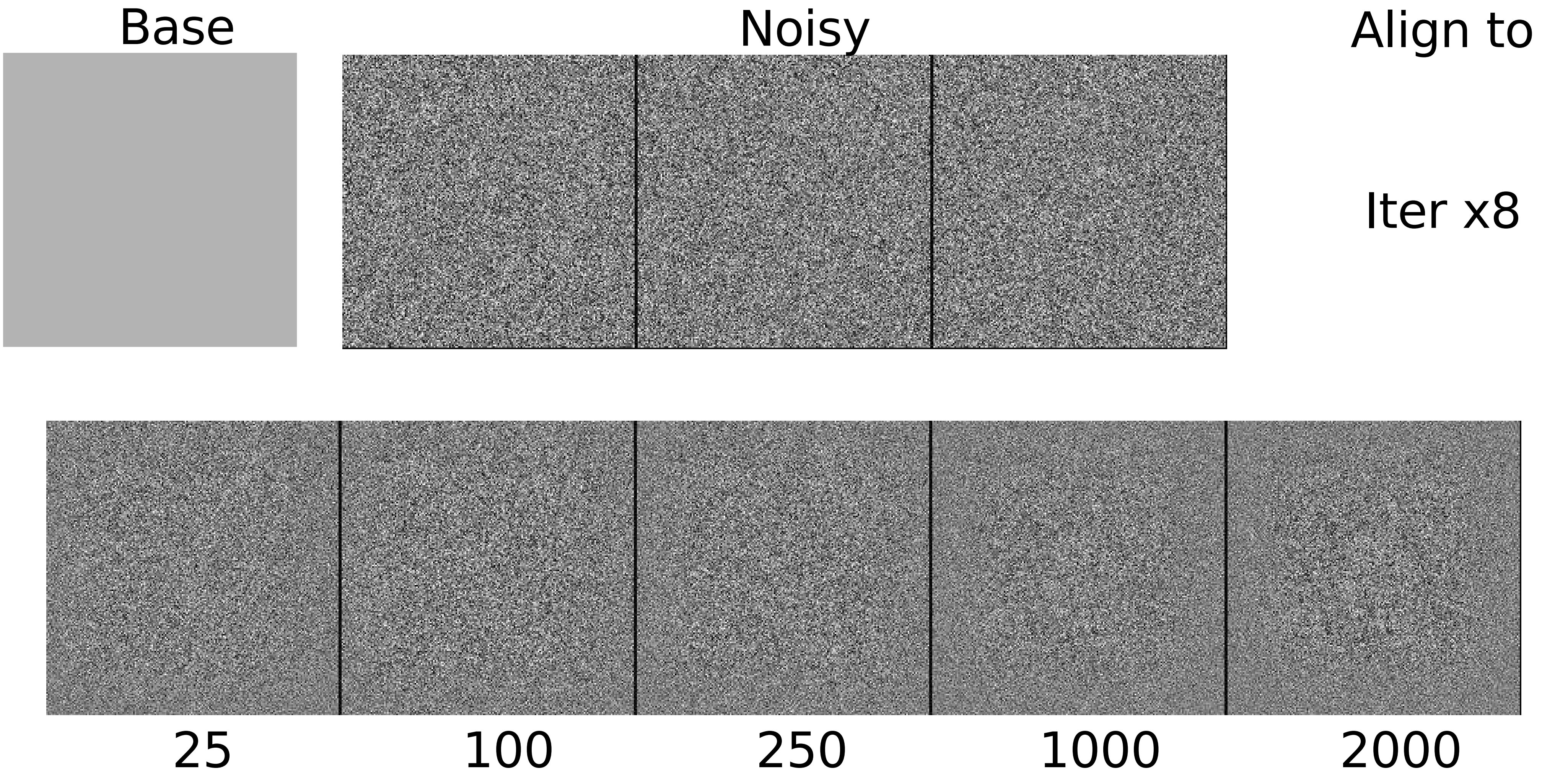
Model Bias



Model Bias



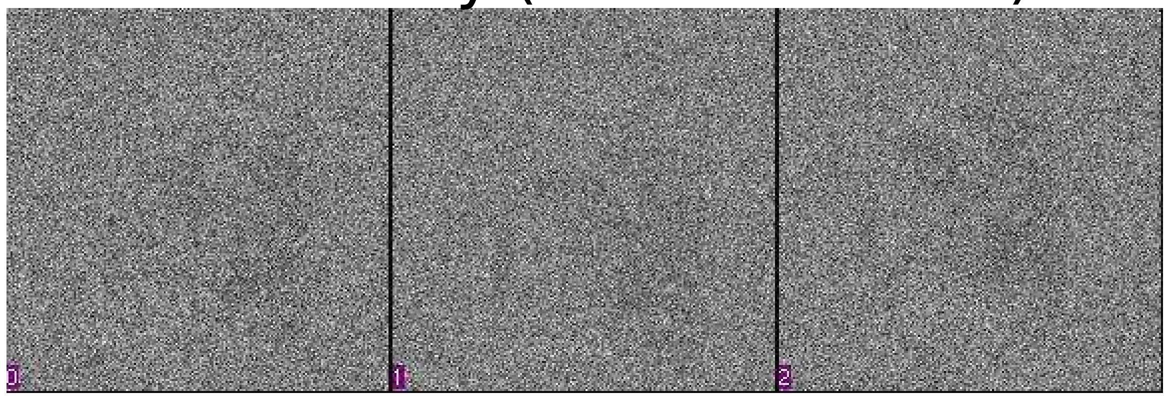
Model Bias



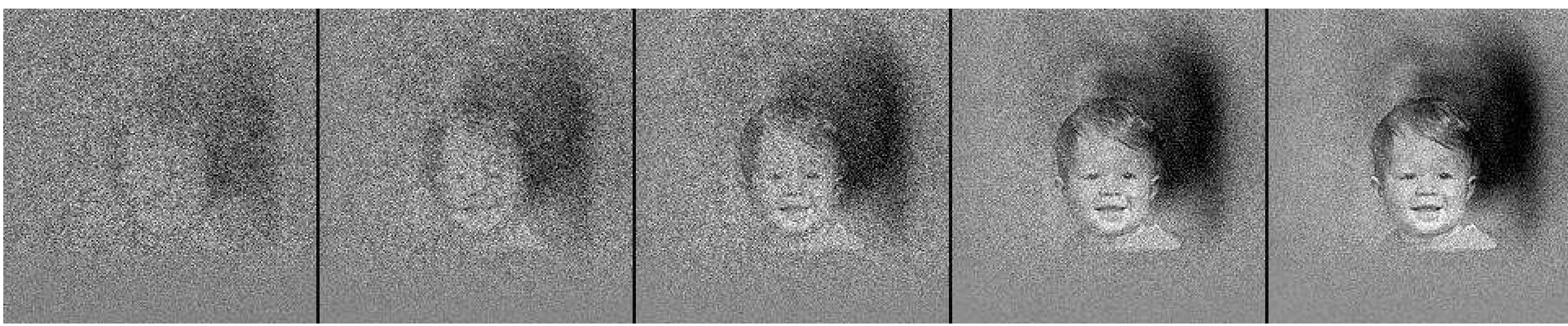
Model Bias

Base

Noisy (~10% contrast)

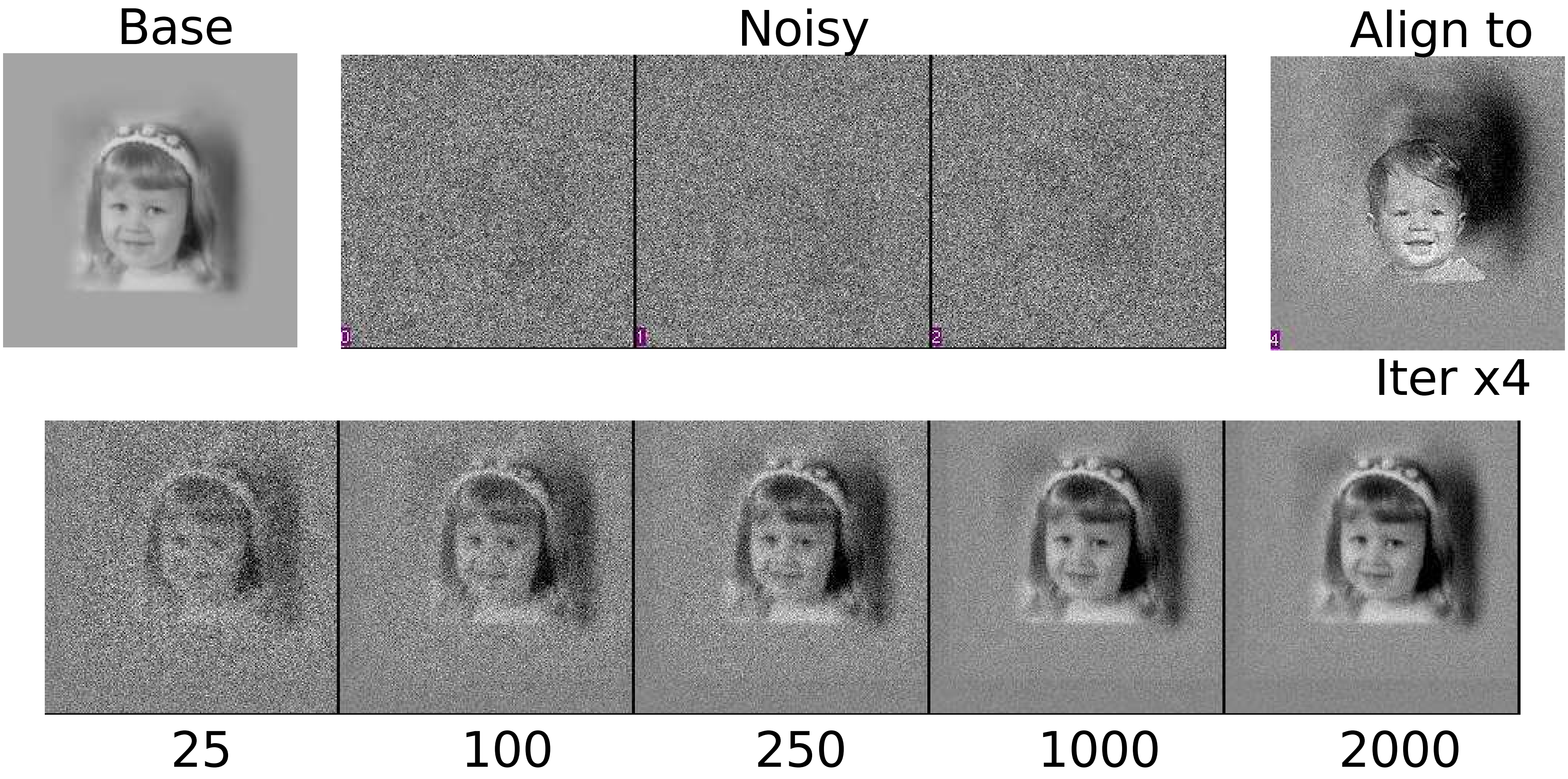


Align to



25 | 100 | 250 | 1000 | 2000

Model Bias



Science of Estimation

Caltech: Ph101, Order of Magnitude Physics

- How to make estimates.
- How to decide what physical effects are important in a given situation, or to understand how some system works the way it does.
- How to decide what terms in complicated equations can be omitted or simplified.
- How to figure out the general features of the solutions to equations, without actually solving the equations.

http://www.its.caltech.edu/~oom/homework.html

Unit Analysis

acceleration -> m/s^2
speed -> m/s

At 9.8 m/s^2, how long does it take to achieve a speed of 60 mi/hr ?

At 9.8 m/s^2, how long does it take to go 1000 m ?

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(1 mile = 1600 m, 1 hr = 3600 sec)

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actually t=sqrt(2d/a)

From Ph101

There are about 100,000 pieces of space junk (exploded satellite fragments, trash left by astronauts, etc.) larger than 1 cm orbiting the earth at altitudes between 200 and 400 km. The orbits are more-or-less randomly oriented. At orbital velocities, a 1 cm marble can penetrate and disable Space Station Freedom. What is the probability that it will be disabled by a collision during the next decade?

From Ph101

The radius of the earth is $6x10^8$ cm, a 200 km = $2x10^7$ cm can be ignored. ie – orbital velocity at 200 km ~ orbital velocity at the surface.

v=sqrt(G M/r) ~ $8x10^5$ cm/sec

Cross-section of space-station ~ size of house ~100 m^2 = 10^6 cm^2

Volume of 200 km orbital space ~ 4 pi r^2 * $2x10^6$ cm = $9x10^{24}$ cm^3 -> $1x10^{-20}$ obj/cm^3.

station covers $8x10^5$ cm/sec * 10^6 cm^2 = $8x10^{11}$ cm^3/sec

decade = $3x10^8$ sec -> ~2 collisions/decade

From Ph101

If all the manure from all the animals consumed as food (cattle, chickens, swine, etc.) in the US during the past century were spread over the surface area of the US, how thick would the layer be?

From Ph101

The US Government recommended food pyramid recomends 140-240 g of meat, eggs, beans and nuts. So take an average of 200 g of meat per day. Over the past century, there has been an average of about 200 million people in the US.

200g/day-person * $2x10^8$ people * 365 days/yr * 100 years = $1.5x10^{15}$ g

... Our inferred factor is in fact quite close to the empirically well-measured factor of 28 used in setting design and legal requirements for cattle farms. ie – 28 * food = feces

Poop barely floats so ~1 g/cm^3.
US area ~ 4000 km x 2000 km = $8x10^{16}$ cm^2.

so, (28 * $1.5x10^{15}$ g/(1 g/cm^3 * $8x10^{16}$ cm^2) ~ .5 cm

Biochemists Should Know

1 mole of anything weighs its atomic weight in grams
1 mole of gas ~22.4 liters = 2.24×10^4 cm^3
water ~ 1 gm/cm^3 = 1 gm/ml = 1 kg/l
0 C = 273.16 K, room temp ~25 C = 77 F, LN_2 = 77 K, -196 C
r_{water} ~ 1.9 Å = 1.9×10^{-8}cm
Lipid membrane thickness ~50 Å
Typical Cell ~1-100 microns, prokaryotes ~1-10 microns
Protein mass ~ 117 Da * # residues
alpha-helix: 3.6 residues/turn, rise: 1.5 Å/residue, 2.3 Å radius
beta-sheet: 3.3 Å/residue, ~4.5 Å strand separation

Charged, negative (acidic) hydrophilic: Asp, Glu
Charged, positive (basic), hydrophilic: Lys, Arg, His
Polar, ~hydrophilic: Ser, Thr, Cys, Tyr, Asn, Gln
Nonpolar, hydrophobic: Gly, Ala, Val, Leu, Ile, Met, Phe, Trp, Pro